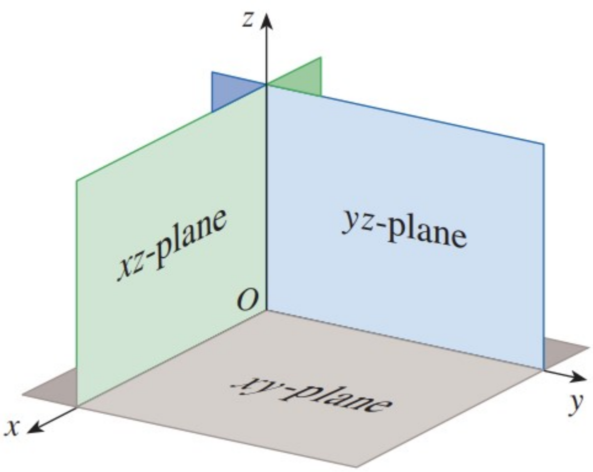
Section 12.1
3D Coordinate Systems

Three-Dimensional Coordinate Systems: What Do They Do? Do They Do Things? Let's Find Out!



Objectives:

- Identify planes, spheres, and cylinders in $\mathbb{R}^3$
- Find the distance between points in $\mathbb{R}^3$

Types of 3D Regions

The **plane** is a region in $\mathbb{R}^3$ of the form, for a, b, c, d in $\mathbb{R}$,

$$ax + by + cz = d.$$

The most common examples of planes are the xy**-plane** (where $z = 0$), the xz**-plane** (where $y = 0$), and the yz**-plane** (where $x = 0$).

Example 1. (Math3D) What is the **projection** of the point $(7, 9, -1)$ on the xz-plane?

Example 2. (Math3D) Sketch the graph of the plane $2x + 4y + 6z = 12$.

The **cylinder** is a region in $\mathbb{R}^3$ of the form

$$(x - h)^2 + (y - k)^2 = r^2.$$

Recall that the same space would be a **circle** in $\mathbb{R}^2$. Since no z-variable is specified the z-coordinate can be any number. This free variable gives the region its "vertical" cylindrical shape.

Example 3. (Math3D) Sketch the graph of $x^2 + y^2 = 1$ in $\mathbb{R}^3$.

A cylinder can open in any direction - even diagonally!

A **sphere** is a region in $\mathbb{R}^3$ of the form

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2.$$

The **center** of the sphere is given by (h, k, l), and the **radius** of the sphere is r. The difference between the sphere and the cylinder is that *the z-distance matters*.

Example 4. (Math3D) Write the sphere $x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$ in standard form.

Example 5. (Math3D) What is the equation of the intersection between the sphere from the previous example and the xy-plane?

Example 6. What is the equation of a sphere centered at $(2, 3, 4)$ that touches the yz-plane?

Distance and Midpoints

As we have seen in the above definitions, **coordinates** in $\mathbb{R}^3$ are of the form (x, y, z). Let's take two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. Using the Pythagorean Theorem a few times tells us that the **distance** between P and Q is given by

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

The **midpoint** of the line segment $\overline{PQ}$ is defined to be

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right).$$

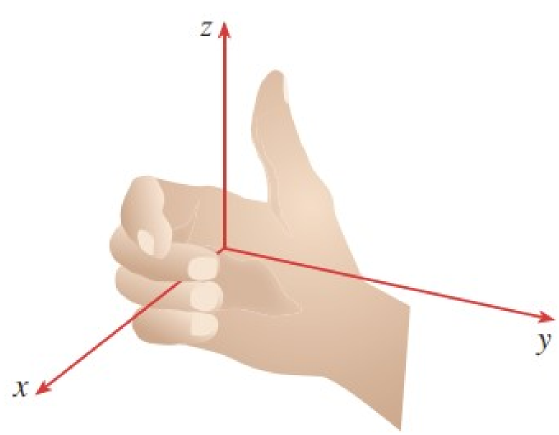
Example 7. Find the equation of a sphere with endpoints $(0, 4, 2)$ and $(6, 2, -2)$.

Right-Hand Rule The direction of the axes in three-dimensional space is given by the *right-hand rule*. You may have already been using this in physics as well - it is useful for determining the positive orientation of an object, which we will discuss in Chapter 16.

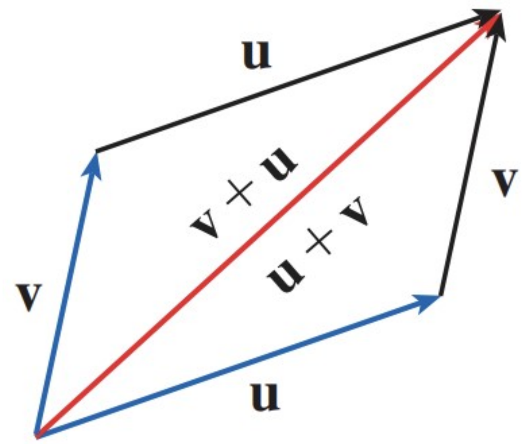
1. Put your fingers in the direction of the first vector.
2. With your fingers pointing this direction, curl them in the direction of the second vector. (You may need to resituate your hand in order to do this.)
3. Your thumb is now pointing in the direction of the third vector.

Try this with the x, y, and z-axes. Point your fingers in the direction of the positive x-axis, then curl it in the direction of the positive y-axis. Which direction is your thumb pointing in?

(This is standard in mathematics, but in aerospace it is common for the positive z-axis to be facing *downward* instead!)



Section 12.2
Vectors in 3D



Objectives:

- Algebraically manipulate 3D vectors
- Identify unit vectors and find unit vectors in the direction of a given vector

A **vector** is a quantity with magnitude and direction. Any coordinate (a, b, c) in $\mathbb{R}^3$ is a vector, and we can think of vectors $\langle a, b, c \rangle$ as arrows with an **initial point** at the origin and a **terminal point** at the coordinate (a, b, c).

NOTE: Unlike coordinates, spatial translates of vectors are considered equivalent.

Example 1. Find the vector **AB** with initial point $A(2, -3, 4)$ and terminal point $B(-2, 1, 1)$.

Algebra of Vectors

Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle, \mathbf{b} = \langle b_1, b_2, b_3 \rangle$ be vectors in $\mathbb{R}^3$ and let c be a **scalar**.

a) **Scalar Multiplication** $c\mathbf{a} = c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$

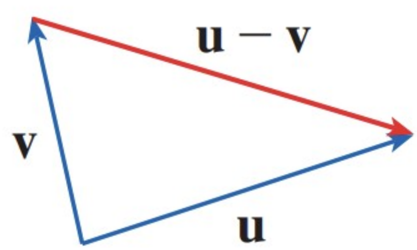
b) **Vector Magnitude** $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

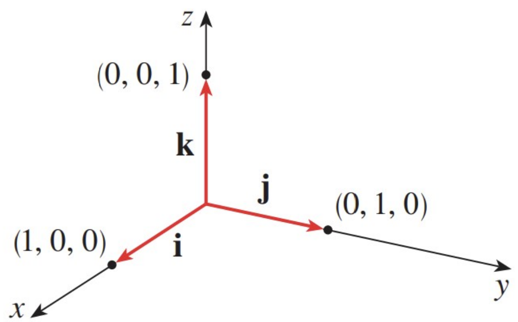
c) **Vector Sum** $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

d) **Vector Difference** $\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$

e) **Unit Vector** The vector $\frac{\mathbf{a}}{|\mathbf{a}|}$ is a **unit vector** of length 1 in the direction of **a**.

f) **Standard Unit Vectors** The vectors **i**, **j**, and **k** are unit vectors in the directions of the positive x, y, and z-axes, respectively.





Example 2. Let $\mathbf{a} = \langle 0, 3, 5 \rangle$ and $\mathbf{b} = \langle 2, 4, 1 \rangle$. Find $\mathbf{a} - 2\mathbf{b}$.

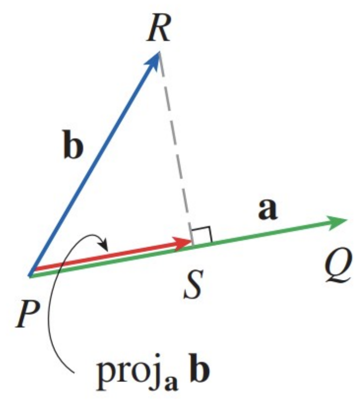
Example 3. Find a unit vector in the same direction as $\langle 3, 4, 12 \rangle$. Then find a vector of length 5 in this direction.

Example 4. Find the following vectors:

i) $\mathbf{i} \times \mathbf{j}$ ii) $\mathbf{i} \times \mathbf{k}$ iii) $\mathbf{j} \times \mathbf{k}$

iv) $\mathbf{j} \times \mathbf{i}$ v) $\mathbf{k} \times \mathbf{i}$ vi) $\mathbf{k} \times \mathbf{j}$

Section 12.3
The Dot Product



Objectives:

- Use the dot product to find the angle between two vectors
- Calculate the projection of one vector onto another

Defining the Dot Product

As simple as the dot product seems, it will find use for us throughout this entire course, especially when two vectors are parallel or orthogonal. We define the **dot product** to be

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta),$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. Note that, for $\mathbf{a}, \mathbf{b} \neq \mathbf{0}$,

$$\mathbf{a} \cdot \mathbf{b} = 0 \quad \Longleftrightarrow \quad \theta = 90^\circ \quad \Longleftrightarrow \quad \mathbf{a} \text{ and } \mathbf{b} \text{ are } \textbf{orthogonal}.$$

The Law of Cosines gives us *another* definition that does not use the angle θ: if $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Example 1. Find the dot product of the vectors $\mathbf{a} = \langle 6, -2, 3 \rangle$ and $\mathbf{b} = \langle 2, 5, -1 \rangle$.

Example 2. Is the expression $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ a scalar, vector, or neither?

Example 3. (Math3D) Find the angle between the vectors $\mathbf{a} = \langle 2, 2, -1 \rangle$ and $\mathbf{b} = \langle 5, -3, 2 \rangle$.

Example 4. How can we tell if two vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel?

Example 5. Let $A(1, -3, -2)$, $B(2, 0, -4)$, and $C(6, -2, -5)$ form a triangle. Find the angle $\angle ABC$.

Vector Projections

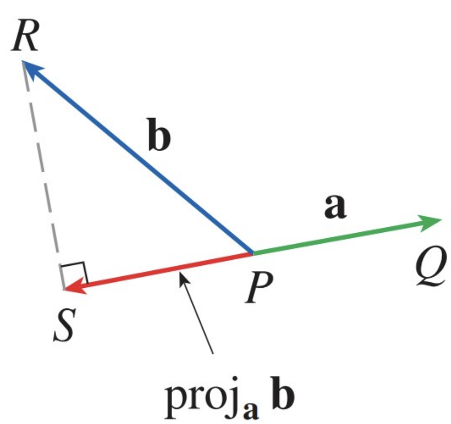
In Example 1 of 12.1 we found the projection of a point onto a plane. This can be thought of as the *shadow* of the point from a light beaming down directly onto the plane beneath. We can think of the **vector projection** $\text{proj}_{\mathbf{a}}\,\mathbf{b}$ of $\mathbf{b}$ onto $\mathbf{a}$ as the *part* of $\mathbf{b}$ in the direction of $\mathbf{a}$. The formula for this is given by

$$\text{proj}_{\mathbf{a}}\,\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\frac{\mathbf{a}}{|\mathbf{a}|} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}.$$

Since the vector projection is in the direction of $\mathbf{a}$, it is a scalar multiple of $\mathbf{a}$ - this is clear from the above formula as well. That scalar multiple of the unit vector in the direction of $\mathbf{a}$ called the **scalar projection** of $\mathbf{b}$ onto $\mathbf{a}$.

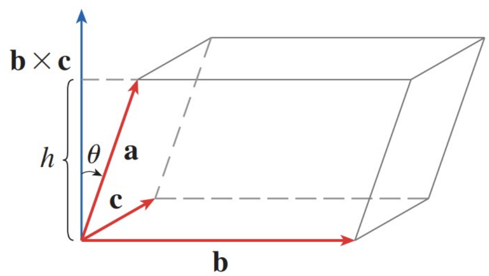
$$\text{comp}_{\mathbf{a}}\,\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}.$$

Sometimes this scalar projection is *negative*, meaning that the vector projection is in the *opposite* direction of $\mathbf{a}$.



Example 6. (Math3D) Find the scalar and vector projection of $\mathbf{b} = \langle 1, 1, 2\rangle$ onto $\mathbf{a} = \langle -2, 3, 1\rangle$.

Section 12.4
The Cross Product



Objectives:

- Calculate the cross product of two vectors using determinants
- Find the area of triangles and parallelograms using vectors

Determinants and the Cross Product

In Section 12.3 we saw that the dot product was not very useful in determining whether two vectors were parallel. It turns out the cross product is one tool that can help us discover this.

The **cross product** of vectors $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ is defined to be

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle.$$

Wow! That's a lot of variables! Is there a mechanism to remember all this by? Indeed, there are a few, and we will choose one that will be useful for us down the road.

A **determinant of order 2** is defined to be

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

A **determinant of order 3** can be defined in terms of order 2 determinants:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}.$$

We can now rewrite the cross product as

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Example 1. (Math3D) Find the cross product of $\mathbf{a} = \langle 1, 3, 4\rangle$ and $\mathbf{b} = \langle 2, 7, -5\rangle$.

Example 2. (Math3D) Show the vector you got from the previous example is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

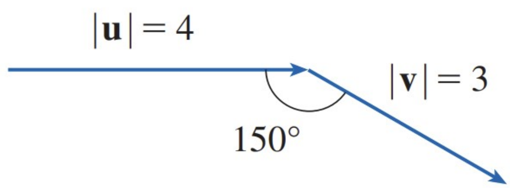
Example 3. Find a vector perpendicular to the plane that passes through the points $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

It can determined directly from properties of the cross product and some calculation that

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin(\theta).$$

NOTE: There is a slight difference in form between this and the similar dot product formula: we have to take the *magnitude* of $\mathbf{a} \times \mathbf{b}$ since, unlike the dot product, $\mathbf{a} \times \mathbf{b}$ is a *vector*, not a scalar.

Example 4. Using the graphic below, find $|\mathbf{u} \times \mathbf{v}|$ and determine its direction.

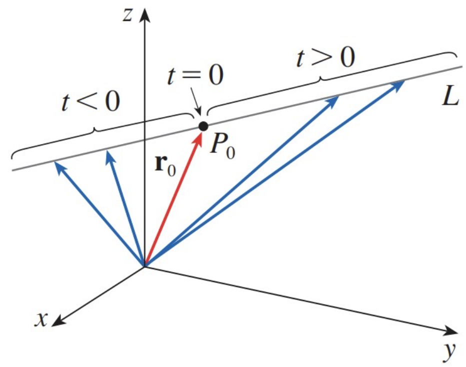


Example 5. Find the area of the parallelogram $PQRS$ passing through the points $P(1, 0, 2)$, $Q(3, 3, 3)$, $R(7, 5, 8)$, and $S(5, 2, 7)$.

Example 6. Find the area of the triangle PQR passing through the points $P(1, 0, 1)$, $Q(-2, 1, 3)$, and $R(4, 2, 5)$.

Section 12.5

Equations of Lines and Planes



Objectives:

- Define lines in $\mathbb{R}^3$ using vectors and parameters
- Write plane equations using their normal vectors
- Find the intersection point between two lines and the angle between two planes

Vectors and Lines in Space

We now know that our standard formula for a *line* in $\mathbb{R}^2$

$$ax + by = c$$

forms a *plane* in $\mathbb{R}^3$. So how do we define lines in $\mathbb{R}^3$? One way is to observe the intersection of two planes, which we will do later. Another way is to introduce the use of a parameter t.

A **parametric equation** of a line passing through point (x_0, y_0, z_0) and parallel to the **direction vector** $\mathbf{v} = \langle a, b, c\rangle$ is given by

$$x = x_0 + at, \qquad y = y_0 + bt, \qquad z = z_0 + ct.$$

If we compile the starting point into a vector $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$ and our running point into $\mathbf{r} = \langle x, y, z\rangle$, then we get the **vector equation** of a line

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + \langle at, bt, ct\rangle;$$
$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}.$$

Example 1. (Math3D) Find the vector equation and parametric equations for the line passing through the point $(5, 1, 3)$ and parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$. Where does this line intersect the xy-plane?

A clever way to eliminate the parameter entirely is to solve for t in each variable of the parametric equation. We can then substitute in each of the other equations in for t to get the **symmetric equations** of a line:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

NOTE: These equations can only be formed if $a, b, c \neq 0$.

Example 2. Find parametric and symmetric equations of the line passing through $A(2, 4, -3)$ and $B(3, -1, 1)$. Where does this line intersect the xy-plane?

Example 3. (Math3D) Find the point of intersection between the lines L_1 and L_2, if there is any, where

$$L_1: \begin{array}{c} x = 1 + t \\ y = -2 + 3t \\ z = 4 - t \end{array} \quad \text{and} \quad L_2: \begin{array}{c} x = 2s \\ y = 3 + s \\ z = -3 + 4s \end{array}.$$

Vectors and Planes

We say a vector $\mathbf{n}$ is **normal** to a plane if it is perpendicular to it. Let us explain why the vector $\mathbf{n} = \langle a, b, c \rangle$ is normal to the plane equation we introduced in Section 12.1

$$ax + by + cz = d.$$

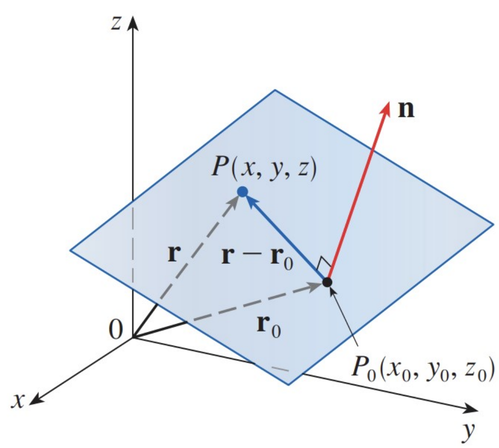
Let $\mathbf{r}_0 = (x_0, y_0, z_0)$ be a point in a plane. Then for any other point $\mathbf{r} = (x, y, z)$ in that plane, the vector $\mathbf{r} - \mathbf{r}_0 = \langle x - x_0, y - y_0, z - z_0 \rangle$ exists entirely in the plane.

Any normal vector $\mathbf{n} = \langle a, b, c \rangle$ would then be perpendicular to this vector; that is,

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \Rightarrow \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0 \Rightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

By expanding the left-hand side and moving constants to the right-hand side, setting the right-hand constant equal to d gets our plane equation. So we write the **equation of the plane** to be

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0.$$



NOTE: The normal vector to a plane is *unique up to scalar multiple*, meaning that all normal vectors to a plane are scalar multiples of each other.

Example 4. Find the plane through the point $(2, 0, 1)$ and perpendicular to the line L_1 : $x = 3t, y = 2 - t, z = 3 + 4t$.

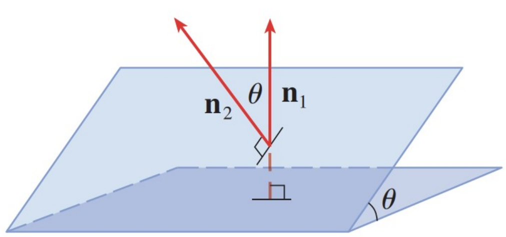
Example 5. (Math3D) Find the equation of the plane through the point $(1, 2, 3)$ and parallel to the plane $x + y + z = 0$.

Example 6. Find the equation of the plane passing through the points $P(1, 3, 2)$, $Q(3, -1, 6)$, and $R(5, 2, 0)$.

Example 7. Find the equation of the plane passing through $(3, 5, -1)$ and containing the line $L_1 : x = 4 - t, y = 2t - 1, z = -3t$.

Example 8. Where does the line $L_1 : x = 2 - 2t, y = 3t, z = 1 + t$ intersect the plane $x + 2y - z = 7$?

Using normal vectors to define a plane has an added advantage: it allows us to find the **angle between two planes** by simply *finding the angle between their normal vectors*. We can show this by using properties of right triangles. We can also see that two planes are **parallel** if and only if their normal vectors are parallel.



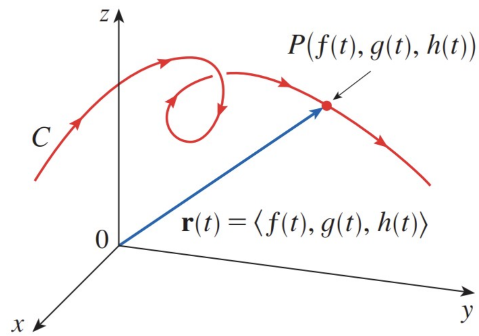
Example 9. (Math3D) Find the angle between the planes $x+y+z=1$ and $x-2y+3z=1$.

We now return to a concept we introduced at the beginning of the section: what region is the intersection of two planes?

Example 10. (Math3D) What region is the intersection of the two planes from the previous example? Write an equation describing it.

Section 13.1

Vector Functions and Space Curves



Objectives:

- Define vector functions and visualize their graphs
- Find domains and limits of vector functions
- Identify space curves with the vector functions they belong to

Vector-Valued Functions

A **vector-valued function**, or **vector function**, is a function from $\mathbb{R}$ to the space of n-dimensional vectors. In this class we will normally be looking at three-dimensional vector vectors, although many of the methods we will use to analyze these can be generalized to any number of dimensions.

We will define vector functions much like how we defined the *vector equation of a line*:

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}.$$

These two forms will be used interchangeably - you may use whichever form you see fit to use in a given scenario.

We will also define many of our pre-calculus and calculus terms in intuitive ways to work with vector functions.

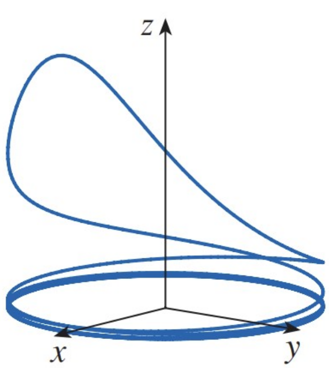
Example 1. Find the **limit** $\lim_{t\to 0} \mathbf{r}(t)$, where $\mathbf{r}(t) = (1+t^3)\mathbf{i} + te^{-t}\mathbf{j} + \dfrac{\sin t}{t}\mathbf{k}$.

Example 2. Find the **domain** of $\mathbf{r}(t) = \langle \ln(t+1), \dfrac{t}{\sqrt{9-t^2}}, 2^t \rangle$.

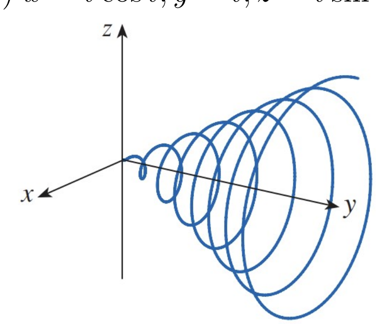
Example 3. At what point(s) does $\mathbf{r}(t) = t\mathbf{i} + (2t - t^2)\mathbf{k}$ intersect the paraboloid $z = x^2 + y^2$?

Example 4. (Math3D) Analyze the following curves:

a) $x = \cos(t),\ y = \sin(t),\ z = \dfrac{1}{1+t^2}$

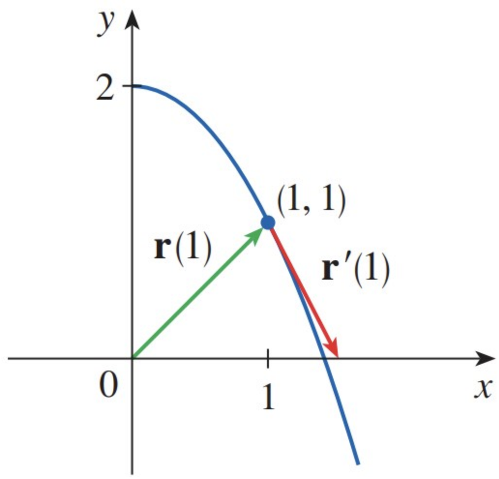


b) $x = t\cos t, y = t, z = t\sin t,\ t \geq 0$



Section 13.2

Derivatives and Integrals of Vector Functions



Objectives:

- Apply derivatives and integrals to vector functions
- Define and calculate the unit tangent vector $\mathbf{T}(t)$

Derivatives

In the spirit of the last section, we continue to define calculus concepts for vector-valued functions. This section will serve as a reminder of the important derivative rules; you must review these concepts in order to succeed in this course.

Example 1. Find the **derivative** $\mathbf{r}'(t)$ if $\mathbf{r}(t) = (\cos(\sin 3t))\mathbf{i} + \left(\dfrac{t^4+1}{t^2+1}\right)^5 \mathbf{j} + \ln(t^2 e^{-1/t})\mathbf{k}$.

Example 2. (Math3D) Find the **unit tangent vector** $\mathbf{T}(0)$, the tengent vector of length 1, to $\mathbf{r}(t) = \langle 2\cos(t), \sin(t), t\rangle$.

Example 3. Find parametric equations for the tangent line of the curve given above at the parameter value $t = \frac{\pi}{2}$.

Example 4. (Math3D) The curves $\mathbf{r}_1(t) = \langle t, t^2, t^3\rangle$ and $\mathbf{r}_2(t) = \langle \sin(t), \sin(2t), t\rangle$ intersect at the origin. Find their angle of intersection.

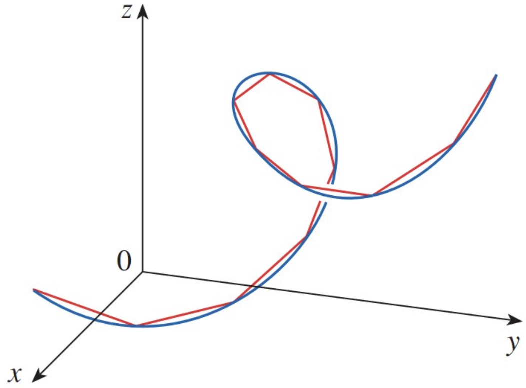
Integrals

It is also imperative to review our integration rules as well as limit rules such as L'Hopital's rule.

Example 5. Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = \langle e^{-t}, \frac{1}{t^2+1}, \frac{1}{2t} \rangle$ and $\mathbf{r}(1) = \langle 1, 1, 1 \rangle$.

Example 6. Find $\int_0^{\frac{\pi}{2}} \mathbf{r}(t)\, dt$ where $\mathbf{r}(t) = 2\cos t\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k}$.

Section 13.3
Arc Length and Curvature



Objectives:

- Determine the length of the space curve defined by a vector function
- Define and calculate the curvature of a curve at a point
- Find the principal unit normal vector

The Arc Length

Recall from Section 10.2 that, in $\mathbb{R}^2$, the length of a parametric curve $(f(t), g(t))$ where $a \le t \le b$ was given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

This generalizes to three dimensions quite nicely and for similar reasons: we still plug the derivatives into the distance function before integrating. The **arc length** of a vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ for $a \le t \le b$ is given by

$$L = \int_a^b |\mathbf{r}'(t)|\, dt = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt.$$

NOTE: We are using $|\cdot|$ to denote distance from 0.

Example 1. Find the arc length of the function $\mathbf{r}(t) = \langle t, 3\cos(t), 3\sin(t)\rangle$ from $0 \leq t \leq 2$.

Example 2. (Math3D) Find the length of the arc of the circular helix with vector equation $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.

Example 3. Find the arc length of the function $\mathbf{r}(t) = \langle t^2, 9t, 4t^{3/2}\rangle$ from $1 \leq t \leq 4$.

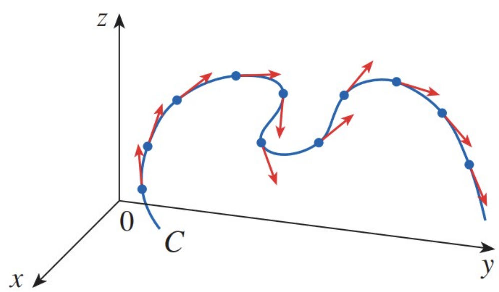
Curvature

Given a curve C, we want a measure of how quickly a curve changes direction without having to refer to a parameter. A curve changing direction would change the direction of the unit tangent vector, so we measure the rate of change of $\mathbf{T}(t)$ with respect to change in s, the distance along the arc itself.

The **curvature** of a curve C is hence defined as

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}/dt}{ds/dt}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}.$$

Re-introducing the parameter t makes it easier to compute this value.



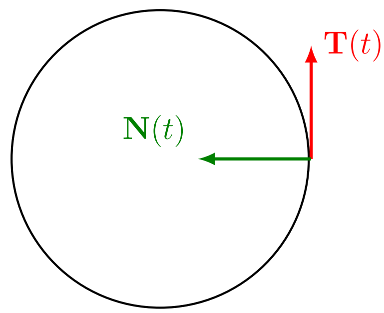
Yet this latter formula is often still more work than it's worth! (Trust me, I tried so you don't have to.) Oftentimes we will use the following curvature formula instead:

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

Example 4. Find the curvature of the twisted cubic $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ at $(0, 0, 0)$.

Principal Unit Normal Vector

Just like we have a canonical tangent vector to a curve, it should make sense to ask for a similar type of normal vector to a curve. (There are actually two - we will only discuss one in this course.)



We define the **principal unit normal vector**

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}.$$

Example 5. (Math3D) Find the unit normal vector for the circular helix $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$.

Section 13.4
Velocity and Acceleration

Objectives:

- Find the velocity, speed, and acceleration of a particle given its position vector function

Velocity and Acceleration

Given a particle moving through space at position $\mathbf{r}(t)$ at time t,

a) The **velocity** of the particle at time t is $\mathbf{v}(t) = \mathbf{r}'(t)$.

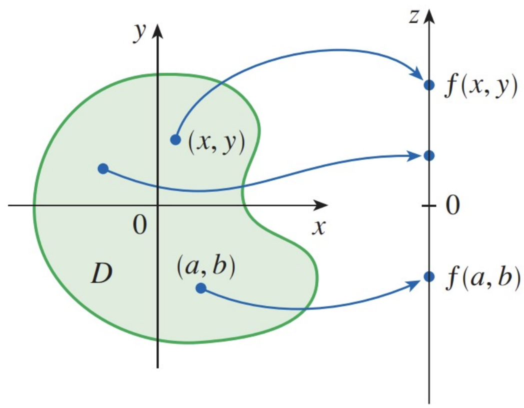
b) The **speed** of the particle at time t is $|\mathbf{v}(t)| = |\mathbf{r}'(t)|$.

c) The **acceleration** of the particle at time t is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

Example 1. The position vector of an object moving in space is given by $\mathbf{r}(t) = \langle t^2, e^t, te^t \rangle$. Find the velocity, acceleration, and speed of the particle at time t.

Example 2. The acceleration of a particle at time t is given by $\mathbf{a}(t) = 2t\mathbf{i} + \sin t\mathbf{j} + \cos(2t)\mathbf{k}$. Knowing $\mathbf{v}(0) = \mathbf{i}$ and $\mathbf{r}(0) = \mathbf{j}$, find the position $\mathbf{r}(t)$ at time t.

Section 14.1
Functions of Several Variables



Objectives:

- Define a function from a domain in two or more dimensions
- Sketch and analyze basic graphs of functions with several variables
- Use level curves and level surfaces to help visualize three-dimensional objects

Functions of Two Variables

In the previous chapter we discussed functions that had vectors as their *output*. Here we will discuss functions with more than one variable in their *input*. Single-variable vector functions can sketch curves; multi-variable functions can sketch *surfaces*.

Example 1. (Math3D) Find and sketch the domain of the following functions:

a) $f(x,y) = \dfrac{\sqrt{x+y+1}}{x-1}$

b) $f(x, y) = x \ln(y^2 - x)$.

c) $f(x, y) = \sqrt{9 - x^2 - y^2}$.

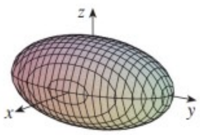
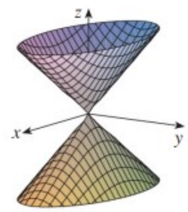
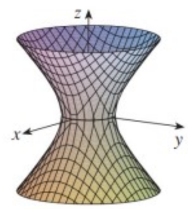
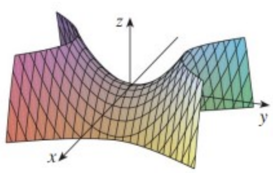
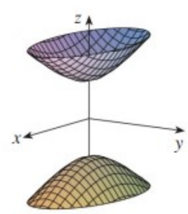
Some common surfaces generated from functions of several variables can be seen in the table on the next page. You may also want to refer back to Section 12.6.

Example 2. Sketch a graph of the following surfaces:

a) $f(x, y) = x^2 + 4y^2 + 1$

b) $f(x, y) = 10 - 4x - 5y$

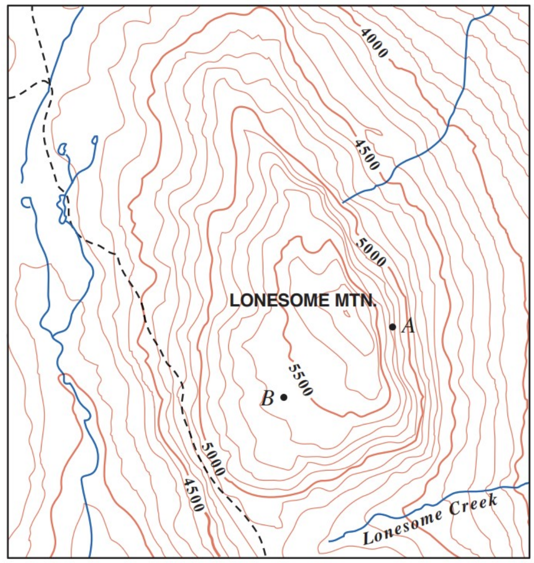
Table 1 Graphs of Quadric Surfaces

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	**Cone** 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	**Hyperboloid of One Sheet** 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	**Hyperboloid of Two Sheets** 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

c) $f(x, y) = y^2$

d) $f(x, y) = \sqrt{x^2 + y^2}$

Level Curves



Another way to visualize three-dimensional graphs in two dimensions is similar to topographical maps (see figure). We can see that the right of the mountain is steep since the lines scrunch together quickly, while the left of the mountain appears to be a more even incline. Each line is called a **level curve** and satisfies the function $f(x, y) = k$ for some constant k, meaning they have the *same* z*-value* (in the figure, z is height). We will call a graph of level curves a **contour map**.

Example 3. Sketch the level curves of the function $f(x, y) = 6 - 3x - 2y$.

Example 4. Sketch the level curves of the function $f(x, y) = 9 - x^2 - y^2$.

Example 5. What type of region(s) are the level curves of the function $f(x, y) = \sqrt{x^2 - y^2}$?

Example 6. What type of region(s) are the level curves of the function $h(x, y) = 4x^2 + y^2 + 1$?

Functions of Three or More Variables

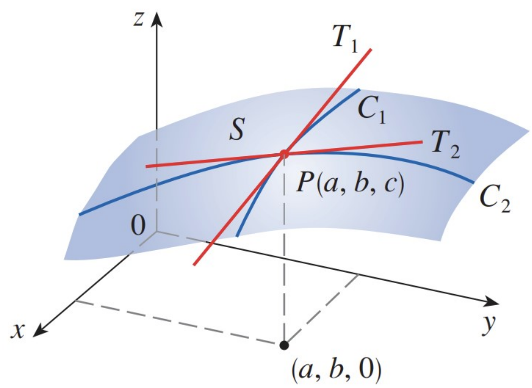
We end this section by mentioning that we can do all of these things in more than just two variables. The issue ends up being that of visualization - with a three-variable function our graph would be in four dimensions. One way to visualize this is by **level surfaces**, which are surfaces of the form $f(x, y, z) = k$ for some constant k.

Example 7. Find the domain of f if $f(x, y, z) = \ln(z - y) + xy \sin z$.

Example 8. Find and analyze the level surfaces of the function $f(x, y, z) = x^2 + y^2 + z^2$.

Example 9. Find and analyze the level surfaces of the function $f(x, y, z) = x^2 - y^2 - z^2$.

Section 14.3
Partial Derivatives



Objectives:

- Define the partial derivative with respect to a variable
- Find first and second order partial derivatives of a function

In Calculus I we took derivatives with respect to the single variable in our function. When dealing with a multi-variable function f there is a similar process. Once we have established a variable to derive with respect to, we *treat all other variables as constants* and then apply the rules from single-variable calculus. This is called the **partial derivative of** f **with respect to a variable**.

Treating one variable constant has the geometric analogue of telling us about "slices" of our function, as one can imagine from our figure for this section.

Example 1. (Math3D) If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

Example 2. If $f(x,y) = \sin\left(\dfrac{x}{1+y}\right)$, calculate $\left.\dfrac{\partial f}{\partial x}\right|_{(x,y)=(\pi,1)}$ and $\left.\dfrac{\partial f}{\partial y}\right|_{(x,y)=(\pi,1)}$.

Example 3. Find $f_x(x,y,z)$, $f_y(x,y,z)$, and $f_z(x,y,z)$ of $f(x,y,z) = e^{xy}\ln z$.

When taking **second partial derivatives**, we simplify the terminology. For example,

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

is the partial derivative of the function f_x with respect to x, and

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

is the partial derivative of the function f_y with respect to x. The functions f_{yy} and f_{xy} are defined similarly.

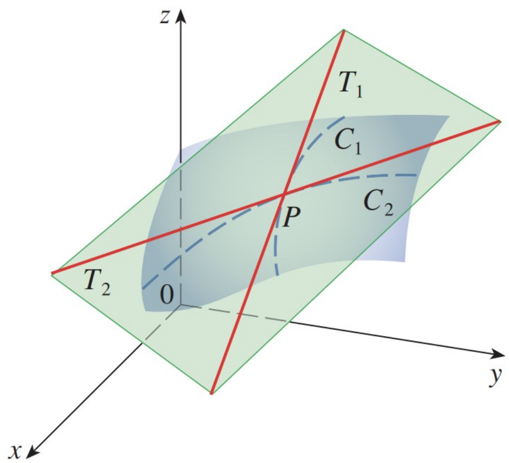
A limit and continuity argument gives us an interesting result known as **Clairaut's Theorem**: if f is defined on a disk $D \in \{(x, y) : x, y \in \mathbb{R}^2\}$ and the **mixed partial derivatives** f_{xy} and f_{yx} are continuous on that disk, then

$$f_{xy} = f_{yx} \quad \text{on } D.$$

Example 4. Find all second-order partial derivatives of the function $f(x, y) = \ln(ax + by)$.

Section 14.4

Tangent Planes and Linear Approximations



Objectives:

- Find the equation of a tangent plane to a surface at a point
- Use differentials to approximate numerical values of functions

Tangent Planes

(Math3D) Now that we have shown how to find tangent lines to a function at a point in the x and y directions, we can define the tangent plane to a function at a point to be the plane containing both of these tangent lines. Say our function is $z = f(x, y)$ and our point is (x_0, y_0, z_0). The formulas for these tangent lines are

$$y = y_0, z = f_x(x_0, y_0)(x - x_0) + z_0 \quad \text{and} \quad x = x_0, z = f_y(x_0, y_0)(y - y_0) + z_0.$$

Combining these two equations into one plane gives us the **equation of the tangent plane** to a surface $z = f(x, y)$ at a point (x_0, y_0, z_0):

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Example 1. Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ when $x = 1$ and $y = 1$.

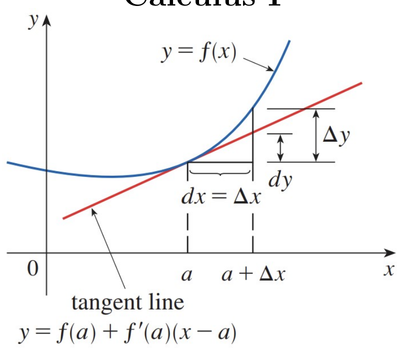
Example 2. (Math3D) Find the equation of the tangent plane to the surface $f(x, y) = e^{x-y}$ at the point $(2, 2, 1)$.

Example 3. Estimate $\sqrt{4.03}$ using differentials.

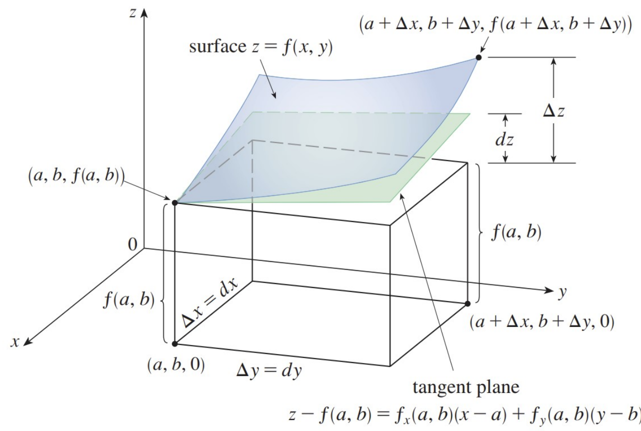
Differentials

Just like how in Calculus I we used differentials to find approximate values of functions close to a fixed point, we have a similar formula for the differential in multi-variable calculus.

Calculus I



Calculus III

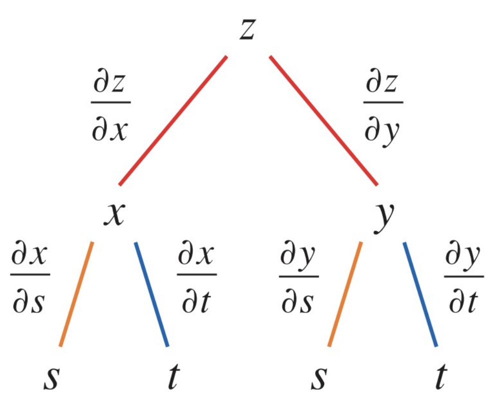


Example 4. For the function $f(x, y) = x^5y^3$, find Δz and dz from $(1, 1)$ to $(1.01, 1.02)$.

Example 5. Approximate $f(x, y) = 1 - xy\cos(\pi y)$ at $(1.02, 0.97)$.

Example 6. Use differentials to estimate the amount of tin in a closed tin can with diameter 8 cm and height 12 cm (measured from the outside) if the tin is 0.04 cm thick.

Section 14.5
The Chain Rule



Objectives:

- Use the chain rule to take derivatives of multi-variable functions whose variables are single variable functions
- Use the chain rule to take derivatives of multi-variable functions whose variables are multi-variable functions
- Solve related rates problems using the chain rule

Chain Rule, Part I

A convenient way to write the chain rule in Calculus I was this: for a function $f(x) = y$ where x is itself a function $x = g(t)$,

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

In multiple variables, this rule stays virtually the same. However, we must take care since, in a function $z = f(x, y)$, *both x and y may be functions of t*. If indeed $x = x(t)$ and $y = y(t)$, it is a calculation using differentials to show that

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

NOTE: The notation for derivatives "d" and partial derivatives "∂" are used with respect to how many independent variables are present in each function. $z = f(x, y)$ has *two* independent variables, so we can write $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. However, since both x and y can be written in terms of t, $z = f(x(t), y(t)) = g(t)$ has only *one* independent variable, hence $\frac{dz}{dt}$.

Example 1. Find $\dfrac{dz}{dt}$ if $z = (\sin 2t)^2(\cos t) + 3(\sin 2t)(\cos t)^4$.

In the problems to come, the intermediate variables (usually x and y) will be specified, but it is always possible to use common elements of a function to make taking its derivative easier by applying the chain rule.

Example 2. Find $\dfrac{dz}{dt}$ if $z = \dfrac{x - y}{x + 2y}$, $x = e^{\pi t}$, and $y = e^{-\pi t}$.

Chain Rule, Part II

The chain rule has even more utility when the intermediate variables (the roles of x and y so far) are functions of *several* variables. The following formulas are useful:

Chain Rule Formulas

If $z = f(x, y)$, $x = g(s, t)$, and $y = h(s, t)$, then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

In general, if $u = f(x_1, x_2, \ldots, x_n)$ and each x_i is a function of $t_1, \ldots, t_m$,

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}.$$

Example 3. If $z = e^x \sin y$, where $x = st^2$ and $y = s^2 t$, find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

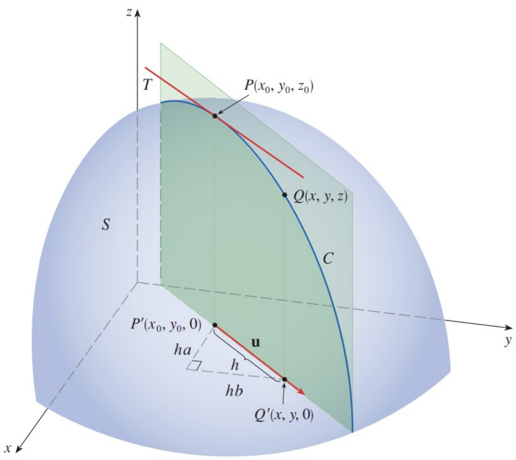
Example 4. Write out the Chain Rule formula for the case of a function $z = f(x, y)$ if $x = g(u, v, w)$ and $y = h(u, v, w)$.

Example 5. If $u = x^4y + y^2z^3$, where $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2s\sin(t)$, find $\left.\dfrac{\partial u}{\partial s}\right|_{(r,s,t)=(2,1,0)}$.

Example 6. The radius of a circular cylinder is decreasing at a rate of 2 cm/s while the height is increasing at a rate of 5 cm/sec. At what rate is the volume of the cylinder changing when the radius is 80 cm and the height is 360cm?

Section 14.6

Directional Derivatives and the Gradient Vector



Objectives:

- Define the directional derivative to a function at a point
- Find the gradient vector and use it to determine direction and magnitude with the greatest rate of change
- Revisit tangent planes with a new definition using the gradient vector

Defining the Directional Derivative

(Desmos, Math3D) Now that we have found the slope of tangent lines to a surface with respect to the x- and y-directions, how can we use these to determine the slope of a tangent line at a point (x_0, y_0, z_0) from a different direction? A great tool at our disposal is the fact that we approach the point from a *line*, so a *linear combination* of $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ ends up doing the trick.

The **directional derivative** of $z = f(x, y)$ at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is

$$D_{\mathbf{u}}f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b = \langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle \cdot \langle a, b\rangle.$$

NOTE: The vector $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$ is so prevalent in Chapter 16 that we will go ahead and define it as the **gradient** of $z = f(x, y)$. ∇f reads as “del f” or “grad f”.

Example 1. If $f(x,y,z) = x^2y + y^2z$, find $D_{\mathbf{u}}f(1,2,3)$ where $\mathbf{u} = \left\langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle$.

Example 2. Find the directional derivative $D_{\mathbf{u}}f(x,y)$ if $f(x,y) = x^3 - 3xy + 4y^2$ and $\mathbf{u}$ is the unit vector given by angle $\theta = \frac{\pi}{6}$.

Example 3. Find the directional derivative of $f(x,y) = e^x \sin y$ at the point $(0, \frac{\pi}{3})$ in the direction of the vector $\mathbf{v} = \langle -6, 8 \rangle$.

Maximum Rate of Change

We now observe the first use of the gradient, and it is a bit surprising. We know the directional derivative is defined as

$$D_{\mathbf{u}}f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle \cdot \mathbf{u} = \nabla f(x_0, y_0) \cdot \mathbf{u},$$

but *what is the maximum value of* $D_{\mathbf{u}}$ *at* (x_0, y_0, z_0)? *In what direction does* f *have the maximum rate of change from* (x_0, y_0, z_0)*?*

For both of these questions, we use the dot product definition to write $D_{\mathbf{u}}f(x_0, y_0) = |\nabla f(x_0, y_0)||\mathbf{u}| \cos\theta = |\nabla f(x_0, y_0)| \cos\theta$.

1. The maximum value is only possible when $\cos\theta = 1 \Rightarrow \theta = 0$, which would mean $\nabla f(x_0, y_0)$ *is in the same direction as* $\mathbf{u}$*.*

2. This maximum value is $|\nabla f(x_0, y_0)| \cdot (1) = |\nabla f(x_0, y_0)|$.

Example 4. What is the minimum value of $D_{\mathbf{u}}f(x_0, y_0)$? In what direction would we have to travel from the starting point?

Example 5. (a) If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q(\frac{1}{2}, 2)$. (b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

Example 6. Find the maximum rate of change of $f(x, y, z) = x\ln(yz)$ at the point $(1, 2, \frac{1}{2})$. Give the direction in which it occurs.

Tangent Planes to Level Surfaces

So far we have been able to find tangent planes to functions of the form $z = f(x, y)$. However, we can sometimes find tangent planes to functions where z's involvement is a bit more complicated. In particular, any parameterized curve $\mathbf{r}(t)$ passing through a level surface of a function of three variables F satisfies the equation

$$F(x(t), y(t), z(t)) = k$$

for some constant k. The chain rule allows us to take the derivative of both sides with respect to t:

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0.$$

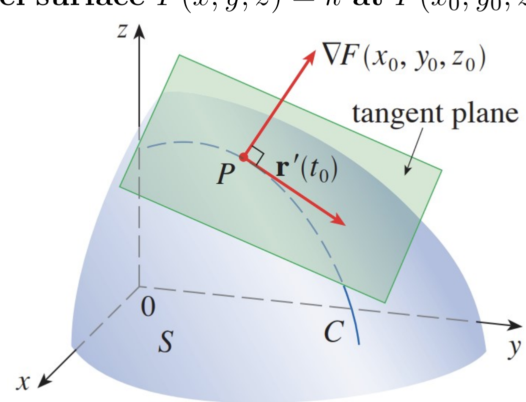
This looks like a dot product waiting to happen. Since $\nabla F = \langle F_x, F_y, F_z\rangle$ and $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle$, we can write

$$\nabla F \cdot \mathbf{r}'(t) = 0.$$

Now we know that *the gradient vector is perpendicular to the tangent vector of our curve.* We do not have to specify x, y, and z here since they already have values once we find t, but we want to be parameter-free. So we write everything in terms of a given point (x_0, y_0, z_0) (now with *any* curve or parameter) and, using our equation of a line, write the **equation of the tangent plane to the level surface** $F(x, y, z) = k$ **at** $P(x_0, y_0, z_0)$ to be

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

From this we see that $\nabla F(x_0, y_0, z_0)$ is the **normal vector** to the tangent plane.



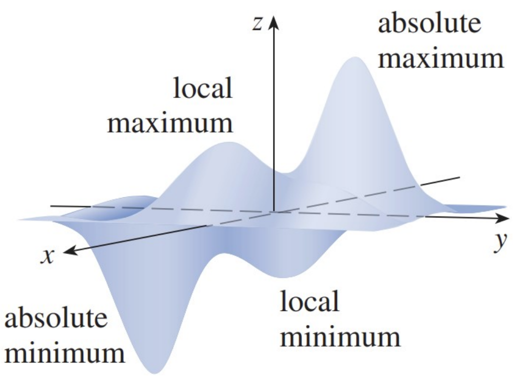
Example 7. Find the equations of the tangent plane and **normal line** at the point $(-2, 1, -3)$ to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3.$$

Example 8. Find the equation of the tangent plane and normal line to the surface $x+y+z = e^{xyz}$ at the point $(0, 0, 1)$.

Section 14.7
Maximum and Minimum Values

...and Where to Find Them



Objectives:

- Use critical points to find local maxima, local minima, and saddle points
- Find absolute maxima and minina on a closed set in $\mathbb{R}^2$
- Utilize the Second Derivatives Test and the Extreme Value Theorem

Local Extrema

We say a function $z = f(x, y)$ has a **local maximum** at $(a, b, f(a, b))$ if $f(x, y) \leq f(a, b)$ for all (x, y) in a small disk around (a, b). We clarify the **local maximum *value*** to be $f(a, b)$ (the z-value only).

We say a function $z = f(x, y)$ has a **local minimum** at $(a, b, f(a, b))$ if $f(x, y) \geq f(a, b)$ for all (x, y) in a small disk around (a, b). We clarify the **local minimum *value*** to be $f(a, b)$ (the z-value only).

From last section we know that the maximum rate of change at a point (x_0, y_0, z_0) is equal to $|\nabla f(x_0, y_0)|$. Hence, if (a, b) is a local maximum (or local minimum, in light of Example 4 from last section), $|\nabla f(x_0, y_0)| = 0$ since the function cannot grow in any direction. So $f_x = 0$ *and* $f_y = 0$; we call points where this happens **critical points**.

Example 1. Find all critical points of $f(x, y) = x^2 + y^2 - 2x - 6y + 14$.

Example 2. Find all critical points of $f(x, y) = (x - y)(1 - xy)$.

Just like in Calculus I, *not all critical points are max/min values.* At some points the function simply rests before continuing its ascent or descent, sometimes in a different direction. We call these **saddle points**.

The **Second Derivatives Test** helps us determine which points are which.

The Second Derivatives Test

Let (a, b) be a critical point of f. Define

$$D = D(a, b) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}.$$

a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $(a, b, f(a, b))$ is a local minimum.

b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $(a, b, f(a, b))$ is a local maximum.

c) If $D < 0$, then $(a, b, f(a, b))$ is a saddle point.

NOTE: If $D = 0$, the Second Derivatives Test gives no information.

Example 3. Find the local maximum and minimum values and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$.

Example 4. (Math3D) Find the local maximum and minimum values and saddle points of $f(x, y) = x^4 - 2x^2 + y^3 - 3y$.

Example 5. A box with no lid is to hold 10 cubic meters. Find the dimensions of the box with a minimum surface area.

Absolute Extrema

We say a set D is **closed** in $\mathbb{R}^2$ if it contains all of its boundary points. An example would be a filled-in disk that contains its boundary: $D = \{(x, y) : x^2 + y^2 \leq 1\}$. We say a set is **bounded** if it is contained within some disk, much like the disk above.

Extreme Value Theorem

If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an **absolute maximum** and **absolute minimum** on D.

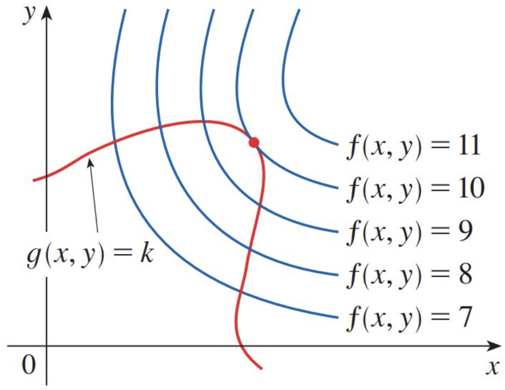
Absolute maxima and minima on D are the highest and lowest point(s) in the set D, respectively. Finding critical points helps us search for any interior point in our set, but *we need to search the boundary* of D to make sure the highest point isn't on the edge. So we follow these instructions:

1. Find the values of the function f at the critical points of f in our set D.
2. Find the highest and lowest values on the boundary of D.
3. The *largest* of the values from steps 1 and 2 is the absolute maximum value; the *smallest* of these values is the absolute minimum value.

Example 6. (Math3D) Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) : 0 \leq x \leq 3, 0 \leq y \leq 2\}$.

Example 7. Find the absolute extrema of $f(x, y) = x^2 + y^2 - 2x$ on the closed triangular region with vertices $(2, 0)$, $(0, 2)$, and $(0, -2)$.

Section 14.8
Langrange Multipliers



Objectives:

- Use Lagrange multipliers to find extreme values on a boundary

In Example 5 of our previous section we were asked to minimize a surface area function subject to a constraint that $V = xyz = 10$. That is, our goal was to **minimize** $A = 2xy + 2xz + 2yz$ **subject to** $V = xyz = 10$. In general, whenever we are asked tot

$$\textbf{Minimize/Maximize } f(x_1, \ldots, x_n)$$
$$\textbf{Subject to: } g(x_1, \ldots, x_n) = k,$$

we can use the method of **Lagrange multipliers** to help find our answer.

In the figure above, say our level curves of $f(x, y)$ are given by the upward-facing curves and our constraint is labeled by $g(x, y) = k$ (in the 2D case). The point seems to be the largest $f(x, y)$ can go on that curve, and we can observe that here *the level curve of f and the space curve g are tangent to each other.* This is because at this point the level curve of $f(x, y)$ touches the level curve *only once*, meaning it cannot grow beyond this level curve.

Since this level curve $f(x, y) = c$ and $g(x, y) = k$ have a common tangent line, their normal lines must be parallel. Since those normal lines each have slope determined by the directions of ∇f and ∇g respectively, we know the *gradient vectors are scalar multiples of each other* at this point - i.e.,

$$\nabla f = \lambda \nabla g \quad \text{for some } \lambda.$$

The number λ is called the **Lagrange Multiplier**.

Although it is harder to visualize, this same logic translates into any larger number of dimensions. Hence we have a new algorithm for how to minimize/maximize functions subject to a constraint:

Method of Lagrange Multipliers

To find the maximum and minimum values of $f(x, y, z)$ on the constraint $g(x, y, z) = k$,

1. Find all values of x, y, z, λ such that

$$\begin{aligned} \nabla f(x, y, z) &= \lambda \nabla g(x, y, z) \text{ and} \\ g(x, y, z) &= k. \end{aligned}$$

2. Evaluate f at all points (x, y, z) that result from step 1. The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Note that the two equations from step 1 actually expand into *four* equations if we separate each coordinate from the first equation into its own separate equation. Hence in many cases we will be able to find a finite number of solutions, and Step 2 will then be possible.

Example 1. Simplify the method of Lagrange multipliers in two dimensions to solving three equations with three unknowns x, y, and λ.

Example 2. Simplify the method of Lagrange multipliers in three dimensions to solving four equations with four unknowns.

Many of these problems require a some ingenuity to solve - we work through several examples below.

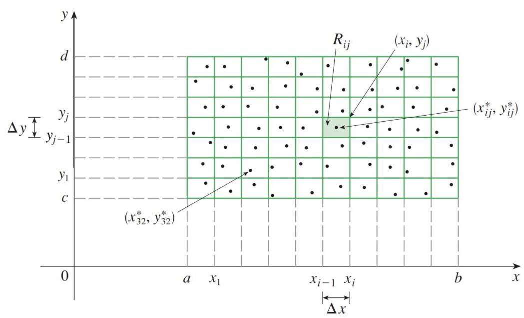
Example 3. (Math3D) Find the extreme values of the functions $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

Example 4. (Math3D) Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$.

Example 5. Find the extreme values of the function $f(x,y) = x^2 - y^2$ on the disk $\{(x,y) : x^2 + y^2 \leq 1\}$.

Example 6. (Math3D) Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex in the plane $x + 2y + 3z = 6$.

Section 15.1
Double Integrals over Rectangles



Objectives:

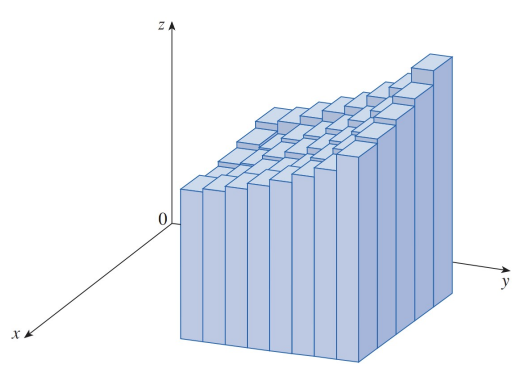
- Find partial integrals of a multi-variable function
- Calculate iterated integrals over an area and use them to find volume of a region

In Calculus I we learned the formula for the definite integral of a function $f(x)$ on an interval $[a, b]$ was given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$ and $x_i^* \in [a + (i-1)\Delta x, a + i\Delta x]$. These subintervals came from dividing the interval $[a, b]$ up into n equal spaces. We then evaluated the function at a point inside each subinterval to get an idea of the height of the function at that interval, and then we shrunk these subintervals down to have length close to 0 to take a limit.

In Calculus III we will do the same thing with small squares on the xy-plane, as you can see in the figure for this section. Here the points (x_{ij}^*, y_{ij}^*) are sample points in each subregion of the function that allow us to approximate the volume underneath with rectangular prisms.



The **double integral** of f over the rectangle R is given by the formula

$$\iint\limits_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A.$$

We will focus less on the formula itself in this course and more on how these subdivisions are made. Note that the areas of the rectangular regions in our above image can be found by calculating $\Delta x \Delta y$, so when we are integrating over a rectangle we will often replace dA with $dx\,dy$ or $dy\,dx$ and integrate accordingly.

When we integrate by either dx or dy, we will integrate *holding the other variable y or x constant*, just like we were when taking derivatives. An integral over a region where one variable is held constant is called a **partial integral**, while the *composition* of these partial integrals like in Examples 1 and 2 is called an **iterated integral**.

Example 1. Find (a) $A(y) := \displaystyle\int_0^2 x + 3x^2y^2\,dx$ and (b) $B(x) := \displaystyle\int_0^3 x + 3x^2y^2\,dy$.

Example 2. Find (a) $\int_0^3 A(y)\,dy$ and $\int_0^2 B(x)\,dx$.

In this chapter we will focus on broadening the type of regions we can integrate over. We will begin with rectangles in this section, move toward polygons and circles, and finally end with regions that can be "transformed" into one of these two.

We found in Examples 1 and 2 that **switching the order of integration** in the problem gave us the same answer. The following theorem helps us know when this is possible:

Fubini's Theorem

If f is continuous on a closed and bounded region R, then switching the order of integration is possible. In particular, if $R = \{(x, y)|a \leq x \leq b, c \leq y \leq d\}$, then

$$\iint\limits_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy.$$

If our function $f(x, y)$ can be separated into the product of two single-variable functions $g(x)h(y)$, then

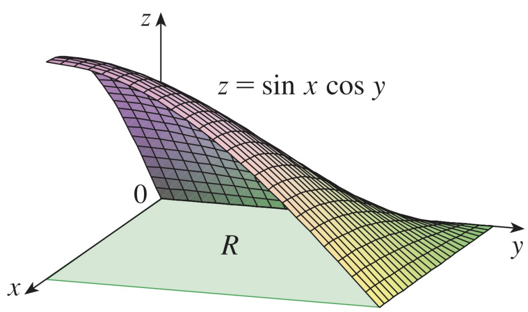
$$\int_a^b \int_c^d g(x)h(y)\, dy\, dx = \int_a^b g(x)\, dx \int_c^d h(y)\, dy = \int_c^d \int_a^b g(x)h(y)\, dx\, dy.$$

Example 3. Evaluate the double integral $\iint_R (x - 3y^2)\, dA$ on the region $R = \{(x, y)|0 \leq x \leq 2, 1 \leq y \leq 2\}$.

Example 4. Evaluate $\iint_R y \sin(xy)\, dA$ where $R = [1, 2] \times [0, \pi]$.

Example 5. Find $\iint_R \sin x \cos y \, dA$ if $R = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$.

Notably, since $\sin x \cos y$ is positive inside this rectangle, this formula gives us the **volume** under the surface within this rectangle.



Example 6. Find $\iint_R ye^{-xy} \, dA$ where $R = [0, 2] \times [0, 3]$.

Recall that a function $f(x)$ is **even** if it is symmetric across the y-axis (i.e., $f(x) = f(-x)$). We also say a function $f(x)$ is **odd** if it is symmetric about the origin (i.e., $f(-x) = -f(x)$). If $f(x) = x^n$, f is an even function if n is even, and f is an odd function if n is odd.

Similarly, we will say that a function $f(x, y)$ is **even/odd in** x if (for even) $f(-x, y) = f(x, y)$ or (for odd) $f(-x, y) = -f(x, y)$. We define **even/odd in** y similarly.

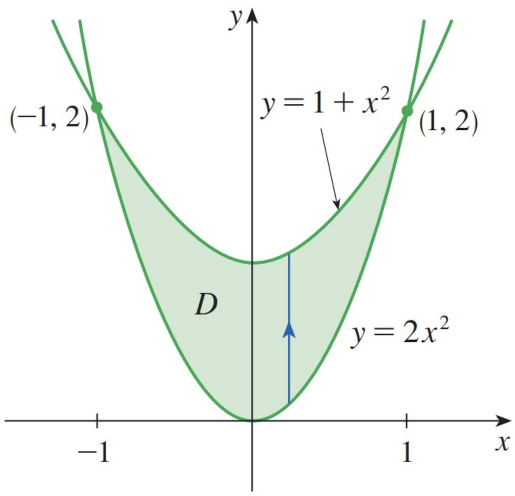
Example 7. (Math3D) Evaluate the integral $\iint_R \frac{xy}{1+x^4}\, dA$ on the rectangle $R = \{(x, y) | -1 \leq x \leq 1, 0 \leq y \leq 1\}$.

Example 8. (Math3D) Find the volume of the solid lying under the elliptic paraboloid $\frac{x^2}{4} + \frac{y^2}{9} + z = 1$ and above the rectangle $R = [-1, 1] \times [-2, 2]$.

Example 9. Find the volume of the solid enclosed by the surface $z = 1 + x^2 y e^y$ and the planes $z = 0$, $x = \pm 1$, $y = 0$, and $y = 1$.

Section 15.2

Double Integrals over General Regions



Objectives:

- Calculate an iterated integral with functions as limits of integration
- Find the volume of regions bounded by general functions

Example 1. Evaluate $\iint_D (x + 2y)\, dA$ where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

As we saw from Example 1, it is possible to set *functions* to be our limits of integration. On a technical level, we are setting our integrand to be equal to 0 outside of our area of integration, then applying the same practices we would when integrating over a rectangle. However, this rarely comes up in practice when solving integrals.

NOTE: Be careful when *switching the order of integration* if there are functions in the limits of integration - we will likely need to change the limits to ensure we are integrating over the same area.

Example 2. Sketch the region D bounded by $y = x$, $y = 4$, and $x = 0$. Then set up, but do not solve, the integral $\iint_D y^2 e^{xy}\, dA$ *in two different ways*. Which one is easier to integrate by?

Example 3. Evaluate the integral $\iint_D xy\,dA$ where D is enclosed by the curves $y = x^2$, $y = 3x$ in *two* different ways.

Example 4. Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$.

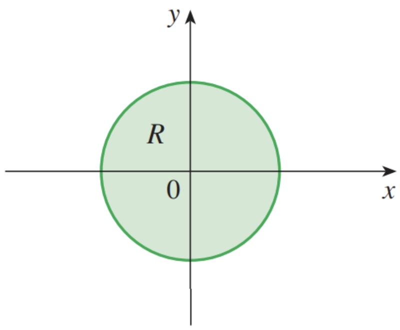
Example 5. (Math3D) Set up, but do not solve, an integral equal to the volume of the region bounded by the coordinate planes and the plane $z = 1 - x - y$.

Example 6. Find the volume of the solid under the plane $3x + 2y - z = 0$ and above the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

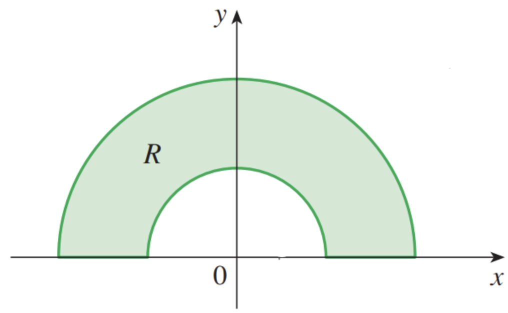
Example 7. Find the volume of the solid in the first octant under the plane $z = x + y$, above the surface $z = xy$, and enclosed by the surfaces $x = 0$, $y = 0$, and $x^2 + y^2 = 4$.

Section 15.3

Double Integrals in Polar Coordinates



(a) $R = \{(r, \theta) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$



(b) $R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$

Objectives:

- Apply a change of variables from rectangular coordinates to polar coordinates
- Find the volume of regions with polar curves tracing out their base

Revisiting Polar Coordinates

At the end of MATH 152 (Section 10.3) we covered polar coordinates and how to graph functions using the variables r and θ. Although it is possible to graph rectilinear areas with polar coordinates, we will usually only use this coordinate system whenever it is easier to refer to *radii and angles* rather than x's and y's.

Example 1. Write the function $(x^2 + y^2)^2 = 2xy$ in polar coordinates.

Converting between Rectangular and Polar Coordinates

$$x = r\cos(\theta),\ y = r\sin(\theta);\ \text{ or } \theta = \arccos\left(\frac{x}{r}\right) = \arcsin\left(\frac{y}{r}\right)$$
$$\tan(\theta) = \frac{y}{x},\ \text{ or } \theta = \arctan\left(\frac{y}{x}\right)$$
$$x^2 + y^2 = r^2,\ \text{ or } r = \sqrt{x^2 + y^2}$$

Example 2. Find the polar coordinates of the given rectangular point if $r \geq 0$ and $0 \leq \theta \leq 2\pi$.

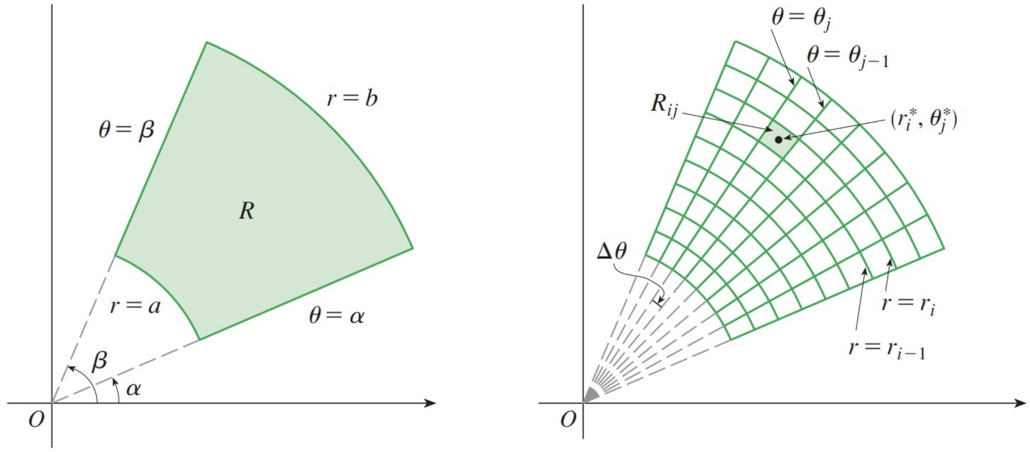
a) $(\sqrt{3}, 1)$

b) $(-\sqrt{3}, 1)$

c) $(-1, -1)$

Example 3. Find the rectangular coordinates of the polar point $\left(2, \frac{2\pi}{3}\right)$.

Integrating over Polar Coordinates



Say we are integrating over a portion of a circle's sector, like shown in the figure for this section. Instead of dividing this area into rectangles, we *radially subdivide* it. Focusing on one of these small regions, we see the width of any of the edges lining up with a radius is $(r_i - r_{i-1})$, where the r_i are radii of the circular arcs making up the region.

Recall that the formula for the *area of a sector* is $A = \frac{1}{2}r^2\theta$, where θ is the change in angle in the sector - in our graphic this is labeled $\Delta\theta$. This region is the *difference* of two sectors, so the area of our region is given by

$$\begin{aligned}
A_{outer\ sector} - A_{inner\ sector} &= \frac{1}{2}r_i^2(\Delta\theta) - \frac{1}{2}r_{i-1}^2(\Delta\theta) \\
\text{Grouping together} \rightarrow &= \frac{1}{2}(r_i^2 - r_{i-1}^2)(\Delta\theta) \\
\text{Difference of squares} \rightarrow &= \frac{1}{2}(r_i - r_{i-1})(r_i + r_{i-1})(\Delta\theta) \\
\text{Grouping together} \rightarrow &= \underbrace{(r_i - r_{i-1})}_{\Delta r}\frac{r_i + r_{i-1}}{2}(\Delta\theta) \\
\text{Replacing factors} \rightarrow &= r_i^*(\Delta r)(\Delta\theta),
\end{aligned}$$

where r_i^* is the average of the two radii r_i and r_{i-1}. At the level of differentials, the two radii r_i and r_{i-1} get closer together, so we can replace r_i^* with r and we get

$$dA = r\,dr\,d\theta.$$

Therefore the **double integral of f over a polar region** D is given by

$$\iint_D f(x,y)\,dA = \iint_D f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

Example 4. (Math3D) Evaluate $\iint\limits_R \frac{y^2}{x^2+y^2}\,dA$, where R is the region that lies between the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Example 5. Find the volume of the solid bounded by the plane $z=0$ and the paraboloid $z=1-x^2-y^2$.

Example 6. Evaluate $\iint_R (2x - y)\, dA$, where R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $y = x$.

Example 7. Evaluate $\iint_D e^{-x^2-y^2}\, dA$, where D is the region bounded by the semicircle $x = \sqrt{4 - y^2}$ and the y-axis.

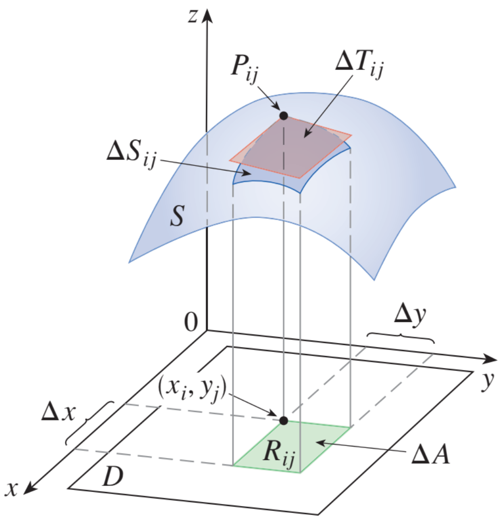
Example 8. (Math3D) Write the integral $\int_0^4 \int_0^{\sqrt{4x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$ in polar coordinates. (You do not need to solve the integral.)

Example 9. Use double integrals to find the area enclosed by one loop of the four-leaved rose $r = \cos(2\theta)$.

Example 10. (Math3D) Find the volume of the region inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 8$.

Section 15.4 & 15.5

Applications of Double Integrals (Center of Mass and Surface Area)

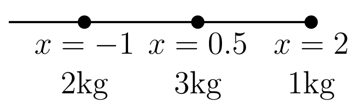


Objectives:

- Calculate mass and center of mass on an object given its density function
- Find the surface area of a three-dimensional region

Center of Mass

Example 1. Find the mass and center of mass for the following discrete system:



Center of Mass

Given a **lamina** occupying a two-dimensional region D and with a density function $\rho(x, y)$ ("$=\frac{\text{mass}}{\text{area}}$"), the **mass** of D is given by

$$m = \iint_D \rho(x, y)\, dA.$$

The **moment about the x-axis** M_x for the lamina D, and respectively the **moment about the y-axis** M_y, is given by

$$M_x = \iint_D y\rho(x, y)\, dA \quad \text{and} \quad M_y = \iint_D x\rho(x, y)\, dA.$$

The **center of mass** $(\overline{x}, \overline{y})$ of a lamina D is given by

$$\overline{x} = \frac{M_y}{m} = \frac{\iint_D x\rho(x, y)\, dA}{\iint_D \rho(x, y)\, dA} \quad \text{and} \quad \overline{y} = \frac{M_x}{m} = \frac{\iint_D y\rho(x, y)\, dA}{\iint_D \rho(x, y)\, dA}.$$

Example 2. Find the mass and center of mass of a triangular lamina with vertices $(0, 0)$, $(1, 0)$, and $(0, 2)$ if the density function is $\rho(x, y) = 1 + 3x + y$.

Example 3. Find the mass and center of mass for the lamina D, where $D := \{(x, y)|1 \leq x \leq 3, 1 \leq y \leq 4\}$ and $\rho(x, y) = y^2$.

Surface Area

Recall from Section 13.3 that the formula for finding the *arc length* of a curve $s(t) = (x(t), y(t), z(t))$ for $a \leq t \leq b$ is given by

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

We can extend this formula to find the surface area of a surface S by turning it into a *double integral* over the integral. We can estimate the surface area of a small region for our surface S as the area of a parallelogram - one edge is determined by the change in x, f_x, and the other by the change in y, f_y.

We know from Section 12.4 that the area of a parallelogram can be determined by finding the length of a cross product, giving us the following formula for surface area (for more details on how this happens, see pages 1026-1027 of our textbook): for a surface S with equation $z = f(x, y)$ for (x, y) in a region D,

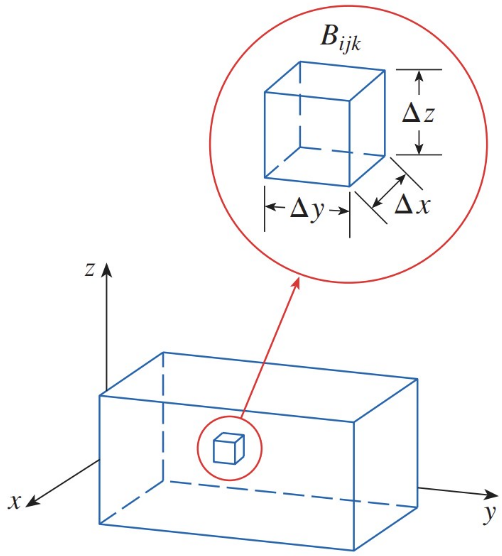
$$A(S) = \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA.$$

Note the similarity to the arc length formula (think $1 = f_z = \frac{\partial f}{\partial z} = \frac{\partial z}{\partial z}$). We will discuss this more in Section 16.6.

Example 4. Find the surface area of the part of the surface $z = x^2 + 2y$ that lies above the triangular region T in the xy-plane with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$.

Example 5. Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

Section 15.6
Triple Integrals



Objectives:

- Calculate iterated integrals with three separate instances of integration
- Find integrals over a volume of space

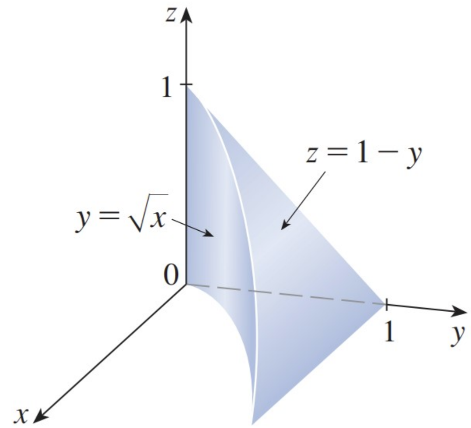
By a similar method to how we extended into three-dimensions in Section 15.1 - integrating a function $z = f(x, y)$ over an xy-region - we can also define an integral in *four dimensions* where we integrate a function $w = f(x, y, z)$ over an xyz-volume. The **triple integral** of a continuous function f over the box $E = \{(x, y, z) : a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$ is

$$\iiint\limits_E f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz.$$

This is a consequence of Fubini's Theorem, which generalizes to three-dimensions, and another consequence is that *switching the order of integration* does not cause any issues over this closed and bounded region as long as we take care to change the limits accordingly.

Example 1. How many ways can we set up a triple integral by switching the orders of integration?

Example 2. (Math3D) Write an integral finding the volume of this region, then rewrite it five other different ways by switching the order of integration.



Example 3. Evaluate $\iiint_E z\,dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

Example 4. Evaluate $\iiint_E \sin y \, dV$, where E lies below the plane $z = x$ and above the triangular region with vertices $(0, 0, 0)$, $(\pi, 0, 0)$, and $(0, \pi, 0)$.

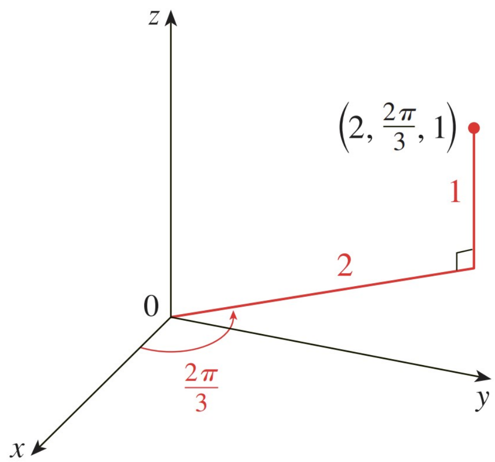
Example 5. (Math3D) Find the value of $\iiint_E z \, dV$, where D is bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$, $y = 3x$, and $z = 0$ in the first octant.

Example 6. Find the integral $\int_0^1 \int_0^1 \int_0^{2-x^2-y^2} xye^z \, dz \, dy \, dx$.

Example 7. Evaluate $\iiint_E \sqrt{x^2+z^2}\, dV$ where E is the region bounded by $y = x^2 + z^2$ and the plane $y = 4$.

Section 15.7

Triple Integrals in Cylindrical Coordinates



Objectives:

- Convert 3-D rectangular coordinates into cylindrical coordinates
- Evaluate triple integrals of functions using cylindrical coordinates

Whenever it seems appropriate to convert *two* of our three coordinates into a more circle-based system than a rectangle-based system, it may be a good idea to switch into using cylindrical coordinates. A good reference for this is the last example in our previous section where we needed to switch to polar coordinates to finish the problem.

A **cylindrical coordinate** is of the form (r, θ, z), where the xy-projection of our point is put into polar coordinates and our z-coordinate stays the same.

Example 1. (a) Find the cylindrical coordinates for the point rectangular point $(3, -3, -7)$.
(b) Find the rectangular coordinates for the cylindrical point $(2, 2\pi/3, 1)$.

Example 2. Write the equations $x^2 - x + y^2 + z^2 = 1$ and $2x^2 + 2y^2 - z^2 = 4$ in cylindrical coordinates.

(Math3D) The formula for integrating in cylindrical coordinates is pretty nifty - in fact, we already integrated with cylindrical coordinates in the last example of the previous section without even knowing it! (Technically, our variables where (r, y, θ) rather than (r, θ, z) since we integrated by y first, but the principle still applies.)

We can find the volume E of the cylindrical sectors like those found in the image below by multiplying the area of the base times the height. Integrating a function $f(x, y, z)$ over z on this box gives us

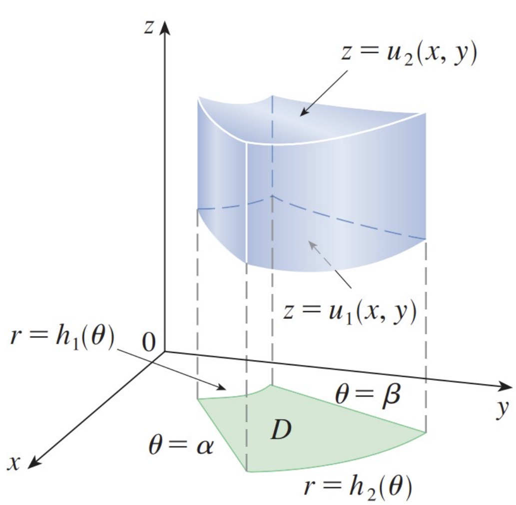
$$\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz,$$

so our volume of the box comes out to be

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz \right] dA.$$

We then only need to integrate by the projection of the area below, which is a sector in polar coordinates! So making the conversion $dA = r\, dr\, d\theta$ that we discussed back in Section 15.3, we finally come to the **triple integral in cylindrical coordinates**:

$$\int_E f(x, y, z)\, dV = \int_E f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta.$$



Example 3. Find the volume of the solid E found with in the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$.

Example 4. Evaluate

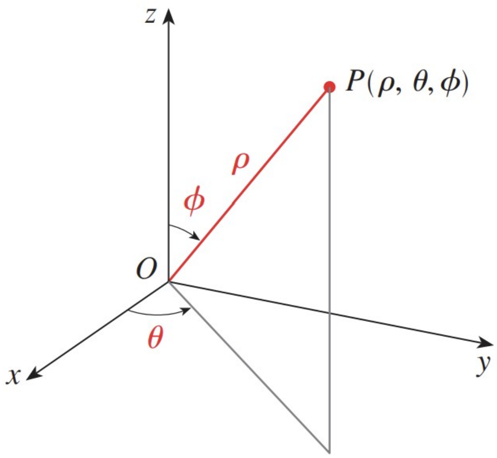
$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2)\, dz\, dy\, dx.$$

Example 5. (Math3D) Set up, but do not solve, an integral equal to the volume of the solid that lies between the paraboloid $z = x^2 + y^2$ and the sphere $x^2 + y^2 + z^2 = 2$.

Example 6. Evaluate $\iiint_E (x - y)\, dV$, where E is the solid that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 16$, above the xy-plane, and below the plane $z = y + 4$.

Section 15.8

Triple Integrals in Spherical Coordinates



Objectives:

- Convert 3-D rectangular coordinates into spherical coordinates
- Evaluate triple integrals of functions using spherical coordinates

Introducing Spherical Coordinates

Even though cylindrical coordinates seem to be a simple modification of polar coordinates to three dimensions, there is maybe one that is more intuitive. Instead of discussing points related to where they are on a cylinder, what about pinpointing where they are on a sphere? A sphere seems the more appropriate three-dimensional analogue of a circle anyway. Although cylindrical coordinates are certainly useful, we will also find lots of use in converting points into spherical coordinates.

The schematic for the **spherical coordinate system** is given above. Here, ρ denotes the radius of the sphere, θ the angle the xy-projection of the point makes with the positive x-axis, and ϕ the angle the point-vector makes with the positive z-axis (in the appropriate plane).

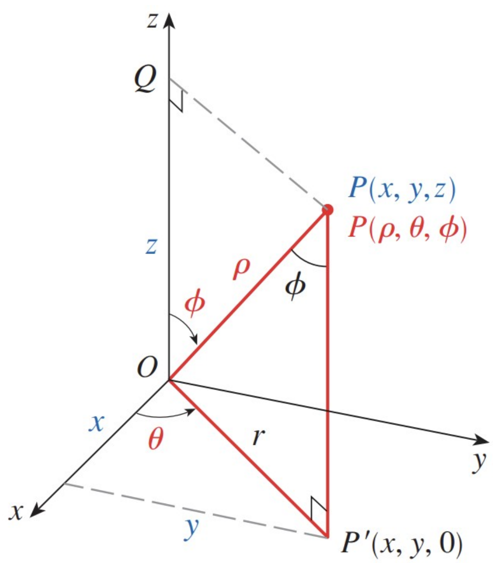
This placement of ϕ might seem strange, since you might expect ϕ to be the angle of elevation for P. In general it is a bit easier to talk about the angle ϕ as coming from a fixed axis: the positive z-axis.

It is also very easy to make duplicate points in this system, so we are careful to restrict $\rho \geq 0$, $0 \leq \theta \leq 2\pi$, and $0 \leq \phi \leq \pi$.

(Math3D) The schematic below helps us find the following formulas for converting into spherical coordinates:

Converting To and From Spherical Coordinates

$$\rho = \sqrt{x^2 + y^2 + z^2}$$
$$z = \rho \cos\phi; \quad r = \rho \sin\phi;$$
$$x = \rho \sin\phi \cos\theta; \quad y = \rho \sin\phi \sin\theta$$
$$\rho \geq 0, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi$$

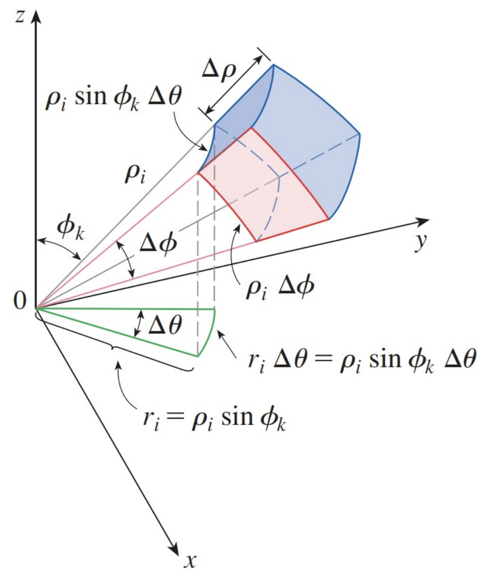


Example 1. Convert $\left(2, \frac{\pi}{4}, \frac{\pi}{3}\right)$ from spherical to rectangular coordinates.

Example 2. Convert $(0, 2\sqrt{3}, -2)$ from rectangular to spherical coordinates.

Example 3. Write the equations $z = \sqrt{x^2 + y^2}$ and $z = x^2 - y^2$ in spherical coordinates.

Integrating over Spherical Coordinates



In previous iterations of this practice where learned how to integrate over a new coordinate system, it was easy to calculate the area or volume of the slice we were integrating. Although we have the tools to do the same here thanks to the Mean Value Theorem (see textbook Exercise 49), we will take a simplistic approach.

If like in previous sections we "spherically subdivide" a space like we did in the above graphic, we see our subregions start to resemble rectangular boxes. Since the volume of these is lwh, it seems worth it to find this volume.

The area of the red "rectangle" is simplest to find: the side closest to the xy-plane beneath it has length $\Delta\rho$ (or $\rho_{i+1} - \rho_i$, as we would use back in Section 15.3), and the side closest to the origin is a circular arc of length changing with the angle ϕ, so given the arc length formula $s = r\phi$ it has length $\rho_i \Delta\phi$.

The remaining side of the box varies with the angle θ, so it is a circular arc as well. But the circle making the arc is a smaller slice from the larger sphere. The radius of that small slice can be found the same way we found r on the previous page, which had formula $\rho \sin\phi$. So using the circular arc formula, this length equals $\rho_i \sin\phi_i (\Delta\phi)$.

Multiplying these together using the volume formula and applying differentials, we find that

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

Therefore, the **triple integral of f over a spherical region** E is given by

$$\iiint_E f(x, y, z) \, dV = \iiint_E f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi) \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

Example 4. Set up, but do not solve, the integral $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$ by converting it into spherical coordinates.

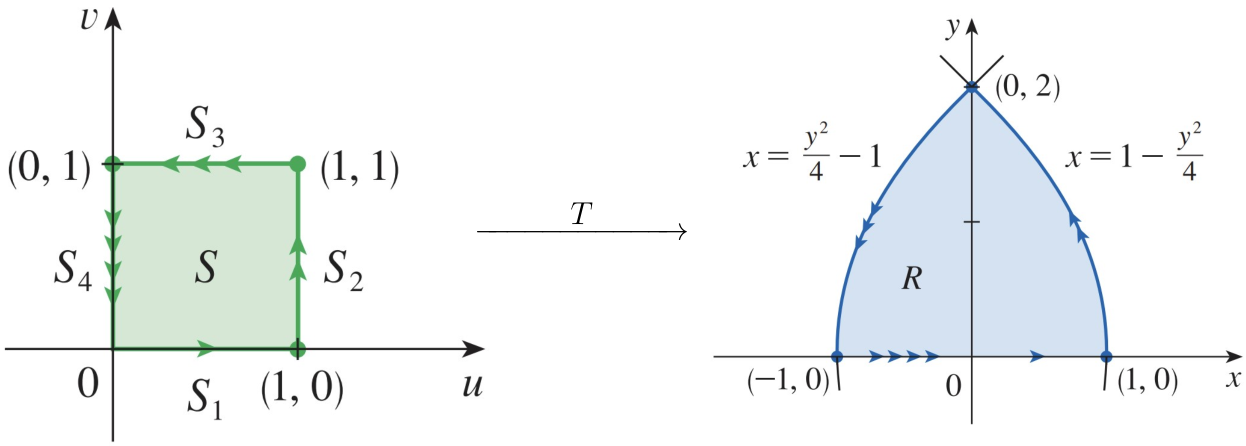
Example 5. Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\,dV$, where B is the unit ball $\{(x,y,z)|x^2+y^2+z^2\leq 1\}$.

Example 6. (Math3D) Find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and inside the sphere $x^2 + y^2 + z^2 = z$.

Example 7. Evaluate the integral $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{2-\sqrt{4-x^2-y^2}}^{2+\sqrt{4-x^2-y^2}} (x^2 + y^2 + z^2)^{3/2} \, dz \, dy \, dx$.

Section 15.9

Change of Variables in Multiple Integrals



Objectives:

- Transform an area we don't know how to integrate over into an area we can integrate over
- Apply the Jacobian of a transformation to solve integrals over a more general realm of areas

The Jacobian

The regions we have integrated over so far this chapter have been convenient for the most part since very few of them have required us to solve multiple different integrals to find one value. In the case that areas of integration become more unruly, we have one final tool up our sleeve: changing the outline of a difficult area to look like an easy area. We have already done this by converting into polar, cylindrical, and spherical coordinates - this is a more general approach.

We define a **transformation** to be a one-to-one function from a region R into a region S. Let's say that our region R is in terms of variables u and v and our region S is in terms of x and y - then if we assume the transformation is *linear* then we only need to see where x and y to see where the whole function goes. The **Jacobian** of a transformation given by $x = g(u, v)$ and $y = h(u, v)$ is given by

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Example 1. Find the Jacobian of the transformation $x = pe^q$ and $y = qe^p$.

Example 2. Find the Jacobian of the transformation from polar to rectangular coordinates.

Changing Variables of Integration

As the last example might suggest, the Jacobian plays a role in replacing variables when integrating. If we are integrating with respect to variables we don't want to integrate by, we can always **apply a change of variables** from region R to region S to get the formula

$$\iint\limits_R f(x, y)\, dA = \iint\limits_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv.$$

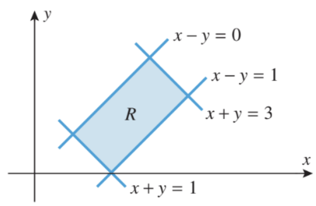
The proof of this formula can be found in pages 1053-1056 of the textbook.

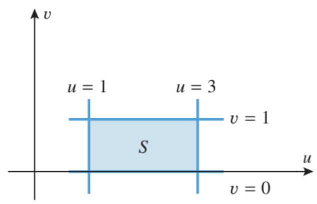
Steps to Apply a Change of Variables

Given a transformation T from a region R in terms of x and y to a region S in terms of u and v, follow these steps to calculate the integral $\iint\limits_R f(x, y)\, dA$:

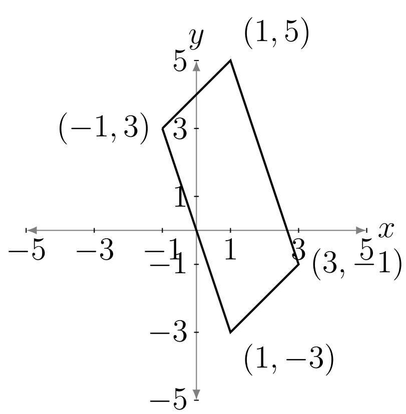
1. Calculate the Jacobian $\dfrac{\partial(x, y)}{\partial(u, v)}$ (derivatives of variables in first region with respect to the second)
2. Use the transformation $T : R \to S$ to find the new area of integration.
3. Apply the variable change to the integrand $f(x(u, v), y(u, v))$ and integrate.

Example 3. Evaluate
$$\iint_R \frac{x-y}{x+y}\, dA,$$
where R is the region enclosed by $x-y=0$, $x-y=1$, $x+y=1$, and $x+y=3$.





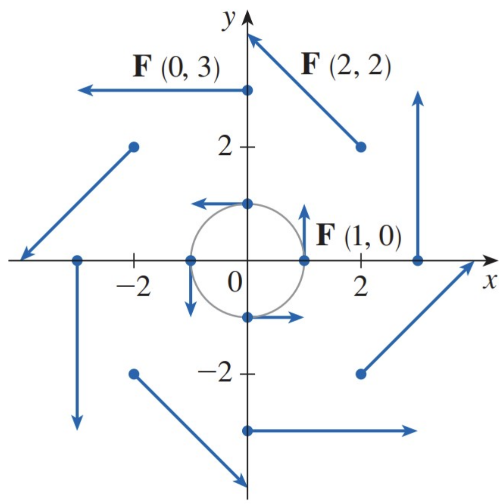
Example 4. Set up, but do not solve, the integral $\iint_R (4x+8y)\, dA$, where R is the parallelogram with vertices $(-1,3)$, $(1,-3)$, $(3,-1)$, and $(1,5)$; applying the change of variables $x=\frac{1}{4}(u+v)$ and $y=\frac{1}{4}(v-3u)$.



Example 5. Evaluate the integral $\iint_R e^{(x+y)/(x-y)}\, dA$, where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-2)$, $(0,-1)$.

Example 6. Evaluate the integral $\iint_R \sin(9x^2 + 4y^2)\, dA$, where R is the region in the first quadrant bounded by the ellipse $9x^2 + 4y^2 = 1$.

Section 16.1
Vector Fields



$$\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$$

Objectives:

- Identify vector fields in their graphical representations
- Revisit the gradient as a vector field

(Math3D) We define a **vector field** in n dimensions to be a function $\mathbf{F}$ that assigns to each point $(x_1, \ldots, x_n)$ in a region $D \subset \mathbb{R}^2$ an n-dimensional vector, $\mathbf{F}(x_1, \ldots, x_n)$. Most physical applications of vector fields are in two and three dimensions - we will keep to those two dimension sizes in this chapter.

Example 1. Find the vector field of gravity.

Example 2. Match the following vector fields with their graphical representations.

I) $\mathbf{F}(x, y) = \langle x, -y \rangle$

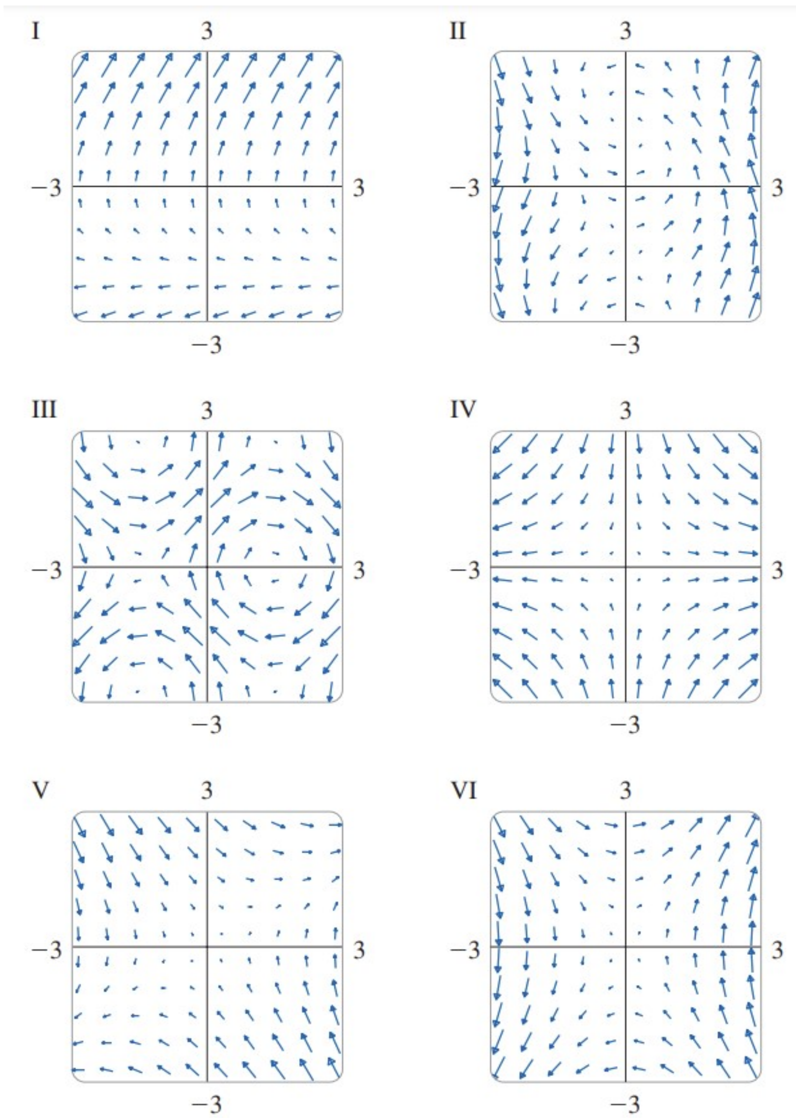
II) $\mathbf{F}(x, y) = \langle y, x - y \rangle$

III) $\mathbf{F}(x, y) = \langle y, y + 2 \rangle$

IV) $\mathbf{F}(x, y) = \langle y, 2x \rangle$

V) $\mathbf{F}(x, y) = \langle \sin y, \cos x \rangle$

VI) $\mathbf{F}(x, y) = \langle \cos(x + y), x \rangle$

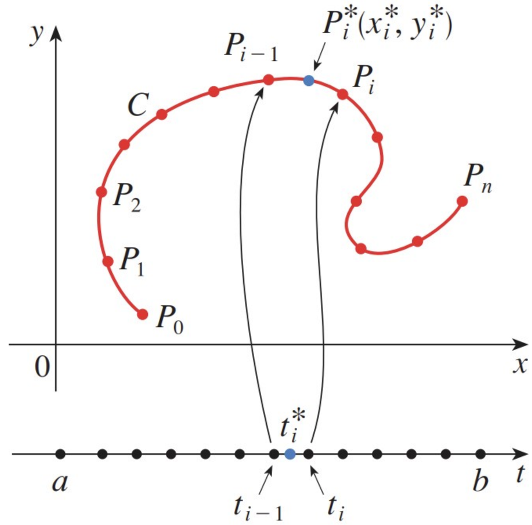


Recall from Chapter 14 that we defined the **gradient** of a function $f(x, y)$ to be $(\nabla f)(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$. Note ∇f assigns a point to a vector - that is the definition of a vector field.

Example 3. (Math3D) Find the gradient vector field of $f(x, y) = \frac{1}{2}(x - y)^2$.

Example 4. Find the gradient vector field of $f(x, y, z) = x^2 y e^{y/z}$.

Section 16.2
Line Integrals



Objectives:

- Calculate line integrals as single-variable integrals over a parameterized space curve
- Apply line integrals to vector fields

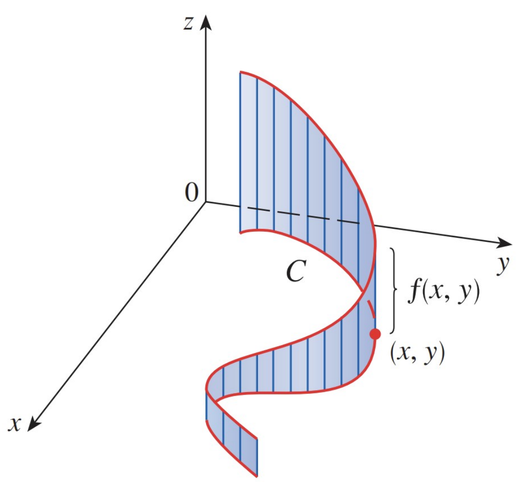
Line Integrals in Space

Example 1. Find the length of the parameterized curve $x = r\cos t$, $y = r\sin t$ from $0 \leq t \leq \theta$.

In single-variable calculus we would find the area under a curve using approximating rectangles. We can do the same single-variable analysis in multiple variables by taking a cross-section of a surface. The result is a space curve that we can integrate with respect to a parameter. This is called a **line integral**.

Example 2. (Math3D) Find the line integral $\int_C x^2 - y^2 \, ds$ where $C : x = t, \, y = 2$ from $0 \leq t \leq 2$.

We do not need to limit ourselves to horizontal or vertical cross-sections; we can take cross-sections with respect to any **piecewise-smooth curve**, such as the shape of the figure for this section:



If we let s denote the arc's length, then the area of our approximating rectangles is given by $f(x_i^*, y_i^*)\Delta s$ (here (x_i^*, y_i^*) is a point chosen in a subinterval on the arc). This is why we wrote the line integral of f along C as $\int_C f(x, y) \, ds$ in Example 2.

However, the length of C can vary a good bit with changes in x and y (unlike the curve in our previous example). So we will often parameterize C in terms of another single variable like t. We know the length of a subinterval of s can be given by the arc length formula we used in Example 1. So the **line integral of f along C** can be written finally as

$$\int_C f(x, y) \, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt = \int_a^b f(x(t), y(t)) |\mathbf{r}'(t)| \, dt.$$

Example 3. Evaluate $\int_C (2+x^2y)\,ds$, where C is the upper half of the unit circle $x^2+y^2=1$.

Example 4. Evaluate $\int_C x^2y+\sin x\,dy$ where C is defined by the vector function $\mathbf{r}(t)=\langle t,t^2\rangle$ from $0\leq t\leq\pi$.

Example 5. Evaluate $\int_C 2x\,ds$, where C consists of the arc C_1 of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$ followed by the vertical line segment C_2 from $(1, 1)$ to $(1, 2)$.

Example 6. Evaluate $\int_C y\,dz + z\,dy + x\,dx$ where $C : x = t^4$, $y = t^3$, $z = t^4$, $0 \leq t \leq 1$.

As we see here, integrating a curve in three dimensions can be easier if the curve is parametrized for us already. However, *we have to be careful about the differentials being used.* For example, integrating a space curve over just the variable z means we are ignoring the curve's motion in any direction other than its projection onto the z-axis.

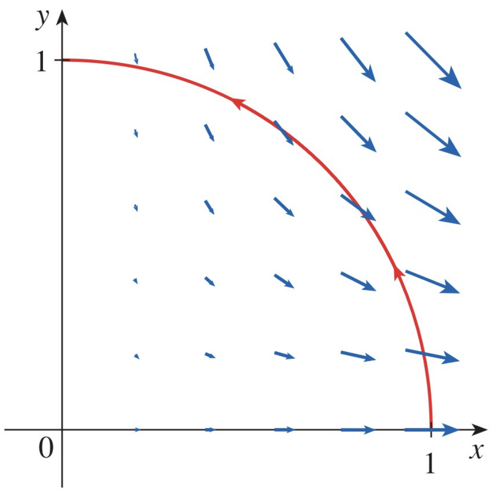
For integrals over s in space, the **line integral of f along C** is what we would expect:

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t))|\mathbf{r}'(t)|\, dt$$
$$= \int_a^b f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

Example 7. Evaluate $\int_C y \sin z\, ds$ where C is the circular helix given by the equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \le t \le 2\pi$.

Line Integrals over Vector Fields

Example 8. Find the work done by the force field $\mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$ in moving a particle along a quarter-circle $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$, $0 \le t \le \pi/2$.



In the previous example we needed to compare directions of the *tangent* to the curve $\mathbf{r}(t)$ to the direction of the force field at the point $\mathbf{F}(\mathbf{r}(t))$. Any *perpendicular* motion came out to be 0; any parallel motion measured the force field's vector there in full. This gives the formula for the **line integral of** F **along** C (also known as **work** done by F on particle C):

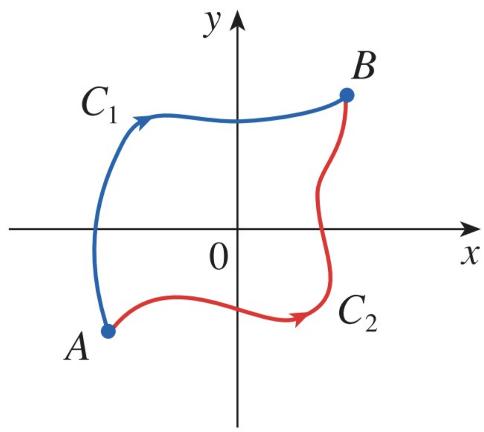
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt.$$

Example 9. Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$ and C is the twisted cubic given by $x = t$, $y = t^2$, $z = t^3$, $0 \le t \le 1$.

Example 10. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle xy^2, -x^2 \rangle$ and C is given by $\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j}$, $0 \leq t \leq 1$.

Section 16.3

The Fundamental Theorem of Line Integrals



Objectives:

- Discover the Fundamental Theorem of Line Integrals as a way to make evaluating line integrals with gradient integrand easier
- Observe line integrals in conservative vector fields

Example 1. For the vector field $\mathbf{F}(x, y) = \langle y^2, x\rangle$, calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

a) C is the curve $y = x^2$ from $(0, 0)$ to $(2, 4)$.

b) C is the line segment connecting $(0, 0)$ to $(2, 4)$.

Our previous example showed a vector field where two different paths between two points yield different line integral values. Recall the difference between this and the Fundamental Theorem of Calculus:

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

In words, this states that the only info we need to find the integral of a function on an area is at the endpoints of its antiderivative. But this was clearly not the case in the previous example; we had the same endpoints but different values in our two parts. The issue is the lack of a clear stand-in for the antiderivative F.

What if we chose $\mathbf{F}(x, y) = \langle y, x\rangle$ instead? If we think about the **gradient** as our derivative, we could ask the question: *what function f has $\langle y, x\rangle$ as its gradient?*

The **Fundamental Theorem for Line Integrals** states that, if $\mathbf{F}(x) = (\nabla f)(x, y)$ for some function f and C is a smooth curve connecting points a and b, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

A corollary (C) is that, if $\mathbf{F} = \nabla f$ for some function f, $\int_C \mathbf{F} \cdot d\mathbf{r}$ is the same value for *any* smooth curve C connecting points a and b. We say such a vector field $\mathbf{F}$ is **conservative**.

Example 2. Determine whether the following vector fields are conservative. If they are, find the function f such that $\mathbf{F} = \nabla f$.

1. $\mathbf{F}(x, y) = (x - y)\mathbf{i} + (x - 2)\mathbf{j}$

2. $\mathbf{F}(x, y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$

The previous example hints at the following result: if $\mathbf{F} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is a vector field such that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},$$

then $\mathbf{F}$ is conservative. This result is a consequence of a theorem we will cover later.

Example 3. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x,y) = (3+2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$, and C is the curve given by $\mathbf{r}(t) = e^t \sin t\mathbf{i} + e^t \cos t\mathbf{j}$, $0 \le t \le \pi$.

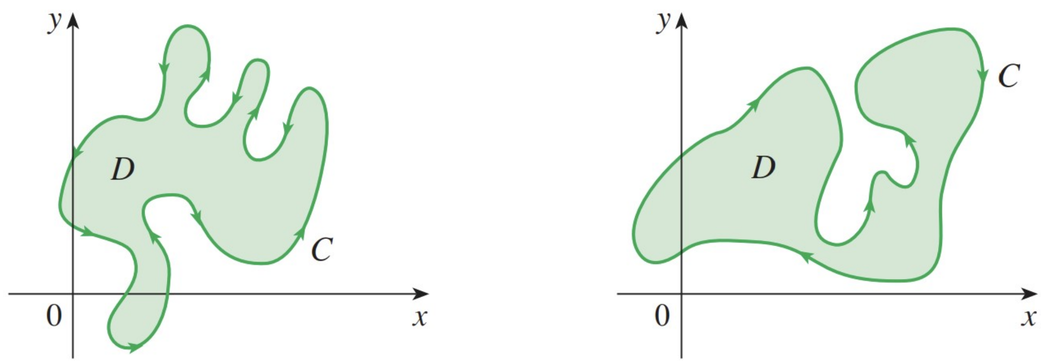
Example 4. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ if $\mathbf{F} = (y^2 z + 2xz^2)\mathbf{i} + 2xyz\mathbf{j} + (xy^2 + 2x^2 z)\mathbf{k}$ and $C : x = \sqrt{t}$, $y = t+1$, $z = t^2$, $0 \le t \le 1$.

Example 5. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = yze^{xz}\mathbf{i} + e^{xz}\mathbf{j} + xye^{xz}\mathbf{k}$ and C is any path connecting $(1, -1, 0)$ and $(5, 3, 0)$.

Example 6. Show the following is equivalent to our corollary (C) above: for any **closed curve** C, meaning a curve connecting a point a back to itself, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

Section 16.4
Green's Theorem

Connecting Line Integrals and Double Integrals



Objectives:

- Apply Green's Theorem to turn line integrals into double integrals or vice versa

While working this next example, consider how solving this line integral over a space curve is similar to how we have been solving line integrals over vector fields. (Compare to the first example of the last section.)

Example 1. Evaluate $\int_C y^2\,dx + x\,dy$, where C is defined by $\mathbf{r}(t) = \langle t, t^2 \rangle$, $0 \le t \le 2$.

As we discussed in the last example of the previous section, a **closed curve** is a curve whose beginning and ending points are the same point. A **simple closed curve** is a curve where the beginning/end point is the *only* place where the curve crosses itself. We will use the symbol $\oint$ to denote a line integral calculated using its **positive orientation**, which is a *counterclockwise* traversal.

Green's Theorem

Let C be a piecewise-smooth simple closed curve and let D be an open region bounded by C. If P and Q have continuous partial derivatives on D, then

$$\oint_C P\,dx + Q\,dy = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

This is *another* fundamental theorem! Compare this with the Fundamental Theorem of Calculus again:

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Green's thoerem says that the *area* over a closed region (in FTC terms: closed *interval*) is determined an "antiderivative" (P and Q vs. $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$) evaluated at its boundary (think *endpoints*). Pages 1097-1098 of our textbook contain a proof that prove this using the Fundamental Theorem of Calculus.

Example 2. Show using Green's Theorem that, for any closed curve C and conservative vector field $\mathbf{F}$, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

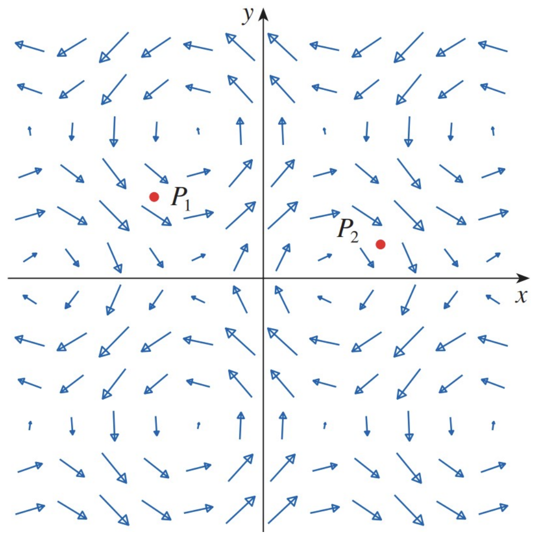
Example 3. Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the triangular curve consisting of the line segments from $(0,0)$ to $(1,1)$, from $(1,1)$ to $(1,0)$, and from $(1,0)$ to $(0,0)$.

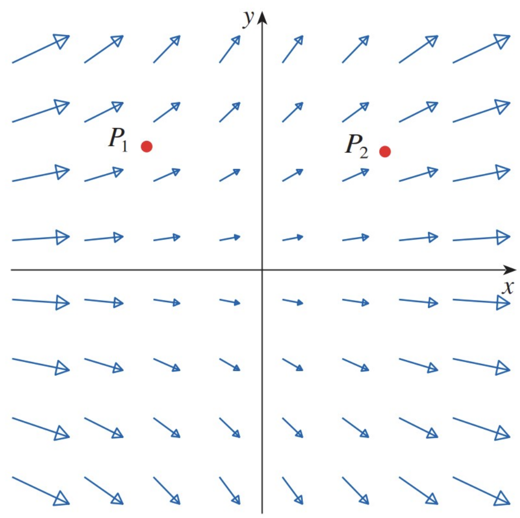
Example 4. Evaluate $\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$, where C is the circle $x^2 + y^2 = 9$.

Example 5. Evaluate $\oint_C y^2\,dx + 3xy\,dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Section 16.5

Curl and Divergence





Objectives:

- Define curl and divergence and the role they play in fluid
- Determine whether three-dimensional vector fields are conservative

Green's Theorem only applies to scenarios in two dimensions. However, as we have seen in previous chapters, most of the definitions and theorems we have learned about generalize to larger numbers of dimensions. We now begin building toward three-dimensional versions of the theorems, picking up some useful definitions along the way.

Vector fields are used in fluid dynamics to measure the flow of water in a space. We say the **curl** of a vector field **F** at a point P is a *vector* quantity measuring the tendency of the fluid to *rotate* in a field around that point. Imagine putting a tiny paddle wheel around the points P_1 and P_2 in the left graphic above - which way would the wheels spin?

We say the **divergence** of a vector field **F** at a point P is a *scalar* quantity measuring the *rate of change* of fluid flowing through the point P. Look at the amount of fluid flowing into points P_1 and P_2 in the right graphic above - is the rate of change positive or negative at each of these points?

We now give formulas for these operators. In both we will use the **del operator** ∇, which is defined as

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle.$$

For example, $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$, which is the gradient of f.

Curl and Divergence

If $\mathbf{F} = \langle P, Q, R \rangle$ is a vector field defined on $\mathbb{R}^3$ and the partial derivatives P, Q, and R exist, then

a) The **divergence** of $\mathbf{F}$ is $\nabla \cdot \mathbf{F} = \left\langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z}$.

b) The **curl** of $\mathbf{F}$ is $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$.

Example 1. Find the curl and divergence of the vector field $\mathbf{F}(x, y, z) = xy^2z^2\mathbf{i} + x^2yz^2\mathbf{j} + x^2y^2z\mathbf{k}$.

As our last example hinted, there is a relationship between *curl* and conservative when it comes to vector fields. Let's say we were to define a two-dimensional version of curl (which, to be clear, we will not call "curl") for a vector field $\mathbf{F} = \langle P, Q \rangle$. What would its formula be?

The analogue of the two-dimensional result also holds:

$$\text{If } \operatorname{curl} \mathbf{F} = \mathbf{0}, \text{ then } \mathbf{F} \text{ is a conservative vector field.}$$

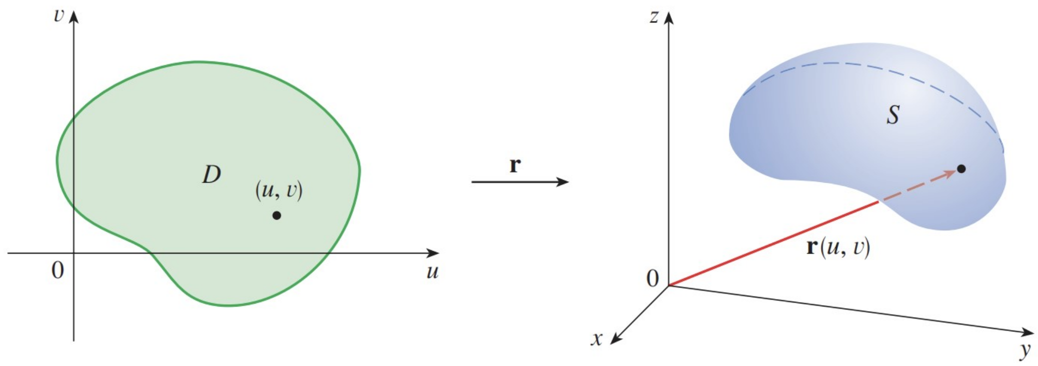
Just as the two-dimensional result requires Green's Theorem, the three-dimensional result requires Stokes' Theorem, a generalization of Green's Theorem for three dimensions!

Example 2. Determine whether $\mathbf{F} = xyz^4\mathbf{i} + x^2z^4\mathbf{j} + 4x^2yz^3\mathbf{k}$ is conservative. If it is, find a function f such that $\mathbf{F} = \nabla f$.

Example 3. If $\mathbf{F} = \langle x, e^y \sin z, e^y \cos z \rangle$, find $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{r}(t) = \langle t^4, t, 2t^2 \rangle$ for $1 \leq t \leq 2$.

Section 16.6

Parametric Surfaces and Their Areas



Objectives:

- Generalize the method of parameterizing curves to surfaces
- Use double integrals to find the area of a wide array of surfaces

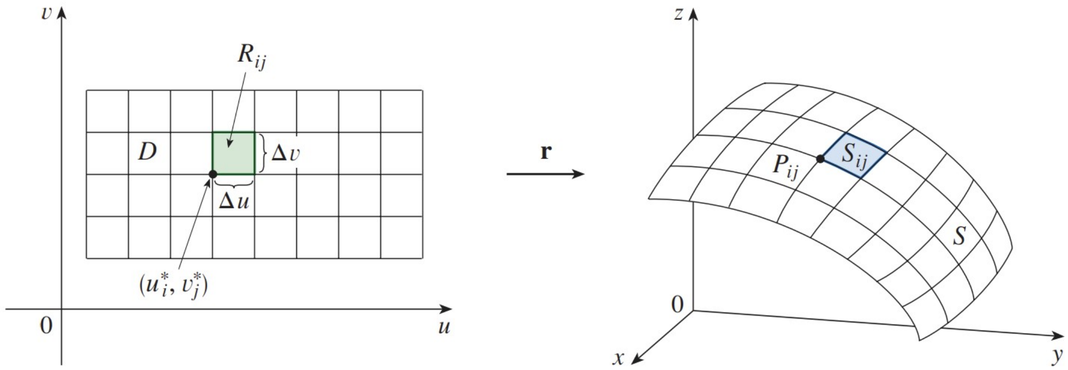
In Chapter 14 we began parameterizing space curves so that we could find arc length. The analogue here is to parameterize surfaces so that we may find surface area. In this case we are given *two parameters* to use rather than one.

Example 1. Parameterize the surface $z + 2x + y = 6$.

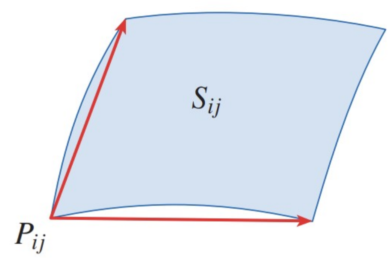
Example 2. Parameterize the cylinder $x^2 + z^2 = 4$ where $0 \leq y \leq 3$.

Example 3. Parameterize the graph $-5y - 4z + x = 20$.

Example 4. Parameterize the part of the sphere $x^2 + y^2 + z^2 = 4$ above the cone $z = \sqrt{x^2 + y^2}$.

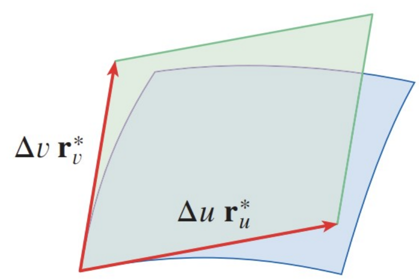


To approximate the area of a surface parameterized in terms of u and v, let's begin dividing it into a series of parallelograms. (Rectangles wouldn't do so well along oblong edges, plus we have a good way to specify what the edges of our parallelograms should be.) We can put our position vector (u_i^*, v_j^*) in the corner of our **patch** (approximating parallelogram) and calculate *tangent vectors* $\mathbf{r}_u^*$ and $\mathbf{r}_v^*$ in the u and v directions.



What's the area of this parallelogram? We know area is $b \cdot h$. If we are given the length of two connecting sides $\mathbf{v}_1$ and $\mathbf{v}_2$ of a parallelogram, we know from Chapter 12 that we can also calculate $|\mathbf{v}_1 \times \mathbf{v}_2|$. The tangent vectors we calculated above already give us the direction of these sides, so we only need to multiply these directions by the scalar length to get $(\Delta u)\mathbf{r}_u^*$ and $(\Delta v)\mathbf{r}_v^*$ respectively.

So we can find the area to be $|(\Delta u \mathbf{r}_u^*) \times (\Delta v \mathbf{r}_v^*)| = |\mathbf{r}_u^* \times \mathbf{r}_v^*| \Delta u \, \Delta v$. As we continue to subdivide, we may replace these incremental values with differentials. The rest is what we did back in Section 15.2: now that we have parameterized our surface, we just need to find the area of our original space, which is $\iint_D 1 \, dA$. So we have our formula:



$$A(S) = \iint\limits_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA,$$

where $\mathbf{r}_u = \dfrac{\partial x}{\partial u}\mathbf{i} + \dfrac{\partial y}{\partial u}\mathbf{j} + \dfrac{\partial z}{\partial u}\mathbf{k}$ and $\mathbf{r}_v = \dfrac{\partial x}{\partial v}\mathbf{i} + \dfrac{\partial y}{\partial v}\mathbf{j} + \dfrac{\partial z}{\partial v}\mathbf{k}$.

(Think of $|\mathbf{r}_u \times \mathbf{r}_v|$ as our "Jacobian" from changing variables.)

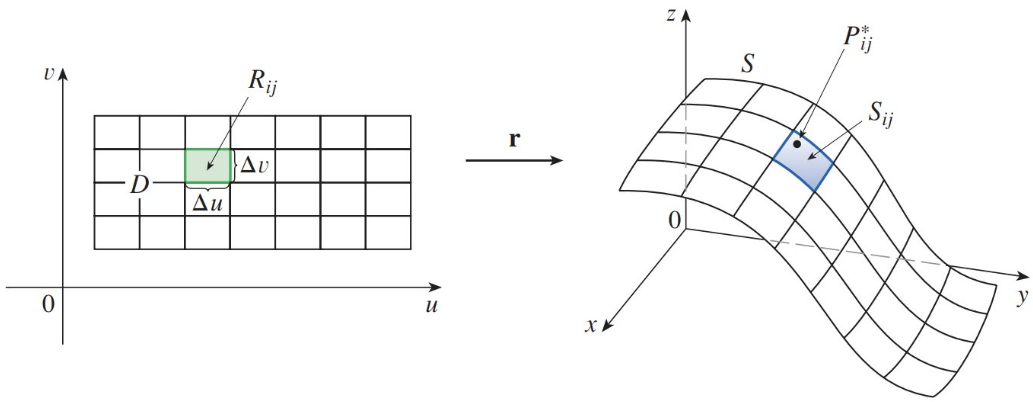
Example 5. (Math3D) Find the area of the part of the surface $x = z^2 + y$ that lies between the planes $y = 0$, $y = z$, and $z = 2$.

Example 6. (Math3D) Find the surface area of the part of the surface $y = x^2 + z^2$ that lies inside the cylinder $x^2 + z^2 = 2$.

Example 7. Find the surface area of a sphere of radius ρ.

Section 16.7

Surface Integrals



Objectives:

- Calculate surface integrals
- Apply surface integrals to double integrals over vector fields

Surface Integrals in Space

A way to write the line integral formula we learned back in Section 16.2 is

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\, dt.$$

Finding the integral over a space curve involved first finding the arc length, then applying the integral structure with the arc length to give appropriate weight to parts of the function where the curve is longer or shorter (with respect to the parameter). The story with surfaces is virtually identical: now that we have found how to find surface area, we just need to apply the integral structure.

By going patch-by-patch, we know the formula for the area of a surface S is $A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA$. Hence the **surface integral of f over S** is given by

$$\iint\limits_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v))|\mathbf{r}_u \times \mathbf{r}_v|\, dA.$$

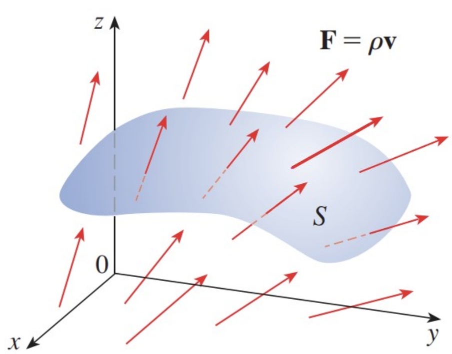
Compare this to the line integral formula written above.

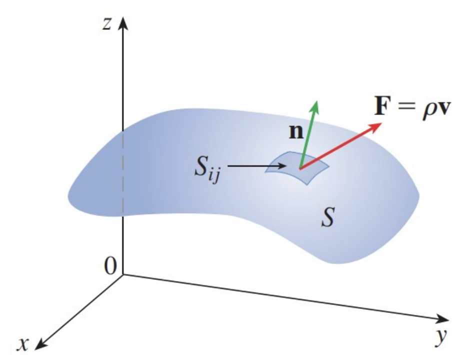
Example 1. Compute the surface integral $\iint_S x^2\, dS$, where S is the unit sphere $x^2+y^2+z^2 = 1$.

Example 2. Evaluate $\iint_S y\, dS$, where S is the surface $z = x + y^2$, $0 \le x \le 1$, $0 \le y \le 2$.

Example 3. Evaluate $\iint_S xyz\,dS$ where S is the cone with parametric equations $x = u\cos v$, $y = u \sin v$, $z = u$, $0 \le u \le 1$, $0 \le v \le \pi/2$.

Surface Integrals over Vector Fields





(Math3D) Suppose $\mathbf{F} = \langle P, Q, R\rangle$ is a vector field that contains surface S. The amount of flow that passes through the surface S is called the **flux** of $\mathbf{F}$ across S. Since we only want to consider the component of $\mathbf{F}$ passing directly through the surface, we take its projection onto the normal vector $\mathbf{n}$. Hence we get the formula

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_D \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,dA.$$

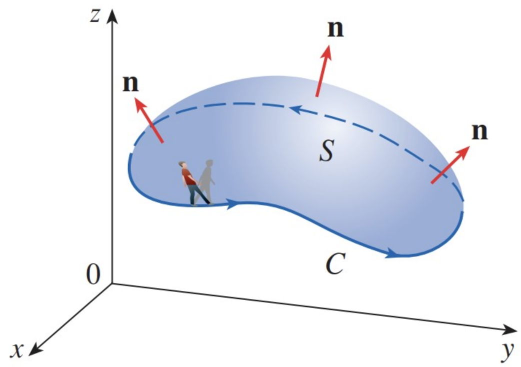
NOTE: This latter integral allows us to avoid some of the hassle of surface integrals. Using properties of the cross product, we can find the vector $\mathbf{n}$ to have the formula $\mathbf{n} = \frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|}$, and the rest comes from our discussion of surface integrals on a previous page.

Example 4. Find the flux of the vector field $\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

Example 5. (Math3D) Find $\iint_S (-x\mathbf{i} - y\mathbf{j} + z^3\mathbf{k}) \cdot d\mathbf{S}$ where S is the part of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 3$ with downward orientation.

Section 16.8 & 16.9

Stokes' Theorem and the Divergence Theorem



Objectives:

- Apply Stokes' Theorem and Divergence Theorem to solve surface integrals involving divergence and curl
- Revisit line integrals and discover how to rewrite them as surface or triple integrals

Stokes' Theorem

Example 1. Back in Section 16.5 we hinted that Stokes' Theorem is a generalization of Green's theorem for three dimensions. Given the tools we have seen so far, guess the formula for Stokes' Theorem.

Stokes' Theorem

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint\limits_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}.$$

A proof of a special case of this result is found in pages 1135-1136 in the textbook.

Example 2. (Math3D) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = -y^2\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$.

Stokes' Theorem works because:

Example 3. (Math3D) Use Stokes' Theorem to compute the integral $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = xz\mathbf{i} + yz\mathbf{j} + xy\mathbf{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane.

Stokes' Theorem works because:

Divergence Theorem

The next theorem covers a class of regions known as **simple solid regions** E. For our purposes, it is enough to know that any *bounded* region - a region that can be contained within a rectangular box - is a simple solid region. Note the boundary of E is a **closed surface** S, meaning that, like a closed curve, the surface divides space into an "inside" and an "outside".

Divergence Theorem

Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV.$$

This theorem states that the *surface integral* of $\mathbf{F}$ across the boundary surface of E is equal to the *triple integral* of the divergence of $\mathbf{F}$ over E. A proof can be found on pages 1141-1143 of our textbook.

Example 4. Find the flux of the vector field $\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

Stokes' or Divergence?

Example 5. (Math3D) Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = xy\mathbf{i} + (y^2 + e^{xz^2})\mathbf{j} + \sin(xy)\mathbf{k}$ and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$.

Stokes' or Divergence?