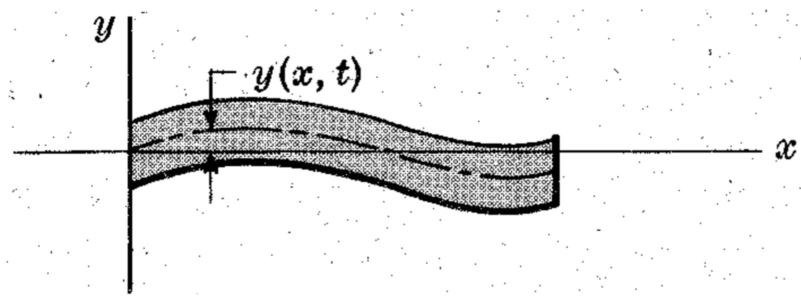
Section S.1

Partial Differential Equations



Objectives:

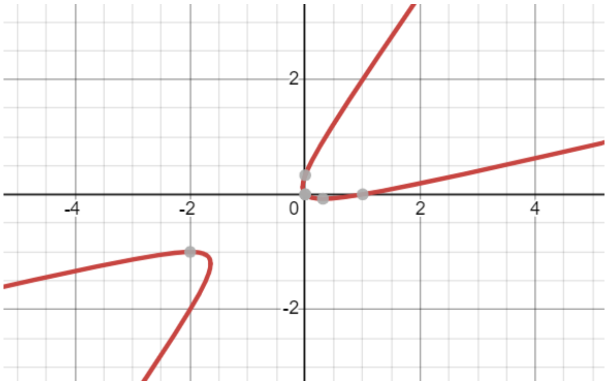
- Solve partial differential equations with boundary conditions using assumptions on the separability of the solution

We now diverge from our discussion of linear algebra to discuss another application of one of our new toys: *Fourier coefficients* (which we already motivated in Section 0-1). Fourier series were discovered by French mathematician and physicist J.B.J. Fourier in attempt to solve a problem regarding the flow of heat in a space given a controlled environment temperature. This discovery sparked the birth of modern analysis, as it turns out to be an interesting question whether Fourier series actually do anything. (We won't get into that here! For us Fourier series will do things. Pinky promise.)

We will assume all definitions from ordinary differential equations and naturally stretch the meaning of a few words as we've learned them. For example, a general **second-order linear partial differential equation** in two independent variables will be of the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G.$$

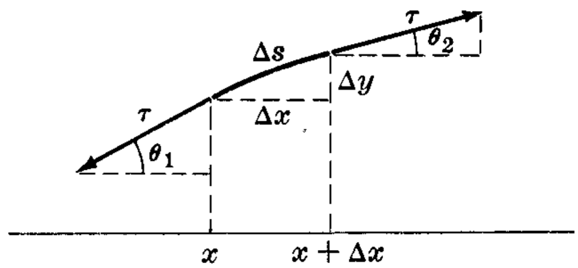
Here the functions A through G may depend on the two independent variables x and y but not on u. (This type of notation is very common - it makes writing the equations a little easier.) If $G = 0$ the equation is **homogeneous**, while if $G \neq 0$ it is called **non-homogeneous**. These equations have **general** and **particular** solutions just as we'd expect from ordinary differential equations.



$x^2 - 5xy + 3y^2 - x - y = 0$

There are a few interesting subclasses of these PDEs depending on whether $B^2 - 4AC$ is greater than, less than, or equal to 0. Informally, the names are based on how these equations look after taking what is known as a Fourier transform. If $B^2 - 4AC > 0$, for example, the PDE is **hyperbolic**. A guide image for this is to the left. If $B^2 - 4AC = 0$, the PDE is **parabolic**, while if $B^2 - 4AC < 0$, the PDE is **elliptic**.

Example S.1.1. Derive a partial differential equation describing a vibrating string with constant tension τ and constant mass per unit length μ.



We will assume small vibrations on the string in this image. So the net upward vertical force acting on Δs is

$$\tau \sin\theta_2 - \tau \sin\theta_1.$$

We will make an approximation that $\sin\theta \approx \tan\theta$ since θ_1, θ_2 are small so that we may introduce the derivative:

$$\sum F = \tau \left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \tau \left.\frac{\partial y}{\partial x}\right|_{x}.$$

By Newton's Second Law tells us that this sum of forces equals the mass of the string, $\mu\Delta s$, times the acceleration of the string, which is $\dfrac{\partial^2 y}{\partial t^2} + \varepsilon$ (since Δs isn't zero, other forces are at play, but if $\Delta s \to 0$ then $\varepsilon \to 0$).

Since our vibrations are small, $\Delta s \approx \Delta x$, so if we divide both sides by $\mu\Delta s$ we get

$$\frac{\tau}{\mu} \frac{\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}}{\Delta x} = \frac{\partial^2 y}{\partial t^2} + \varepsilon.$$

Taking the limit as $\Delta x \to 0$ (in which case $\varepsilon \to 0$), we have

$$\frac{\tau}{\mu}\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial^2 y}{\partial t^2} \quad \Longleftrightarrow \quad \boxed{\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}},$$

where in the last step we set $a^2 = \frac{\tau}{\mu}$.

The equation on the previous page is known as the **vibrating string equation**. Indeed, most of our popular partial differential equations come from physics! The vibrating string equation can be generalized into two dimensions using the equation

$$\frac{\partial^2 z}{\partial t^2} = a^2 \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right).$$

Here are a few more equations we will see:

- **Laplace's equation** is given by $\nabla^2 v = 0$. Recall that ∇ is the **del operator** given by $\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$. So this equation written out looks like

  $$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0.$$

- the **heat conduction equation** is given by

  $$\frac{\partial u}{\partial t} = \kappa \nabla^2 u.$$

  Here $u(x, y, z, t)$ is the temperature at position (x, y, z) in a solid at time t. The constant κ is a *diffusivity* constant of heat through the solid.

Note that Laplace's equation is a special case of the heat conduction equation when we assume that $\frac{\partial v}{\partial t} = 0$ in the heat conduction equation. This means the v we are solving for is the **steady-state temperature**, which is the temperature after a long time has elapsed.

Example S.1.2. Show that $u(x, t) = e^{-8t} \sin(2x)$ is a solution to the boundary value problem

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, \quad u(0, t) = u(\pi, t) = 0, \quad u(x, 0) = \sin(2x).$$

Notice that the "boundary" in this problem is given by the endpoints $x = 0$ and $x = \pi$. We are also given an initial value on the time to fix what the value of the function is at $t = 0$.

The **Laplacian** $\nabla^2 u$ is seen in other equations from fields of gravitation or electricity. We record the Laplacian in cylindrical coordinates here since it will come in handy when discussing Bessel functions in Section S-3:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{\partial r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}.$$

In the study of ordinary differential equations, we are accustomed to not making many assumptions on our solutions in order to determine what they are. The story of PDEs is very different; we are looking for *patterns in the solutions we find* so that when we see similar equations we can assume our solution looks very similar. For example, in the second-order linear PDE, if the functions A through F are constants, the general solution of the homogeneous equation is found by assuming that

$$u = e^{ax+by} \qquad \text{for constants } a \text{ and } b,$$

as we will see here.

Example S.1.3. Find solutions of

$$\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial^2 u}{\partial y^2} = 0.$$

Example S.1.4. Solve the PDE $\frac{\partial^2 z}{\partial x\,\partial y} = x^2 y$.

Example S.1.5. Using the solution above, find the particular solution for which $z(x, 0) = x^2$ and $z(1, y) = \cos(y)$.

Example S.1.6. Solve $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 10e^{2x+y}$.

Example S.1.7. Solve $\dfrac{\partial^2 u}{\partial x^2} - 4\dfrac{\partial^2 u}{\partial y^2} = e^{2x+y}$.

One method of solution for boundary value problems is the **separation of variables**, which is assuming that a solution can be expressed as a product of unknown functions, each of which depends on only one of the independent variables. The success of this method hinges on whether plugging in this form of equation into the PDE yields an equation of the form

$$F(x) = G(y)$$

for some functions F and G, in which case both must be equal to a constant.

Example S.1.8. Solve the boundary value problem

$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}, \quad u(0, y) = 8e^{-3y}.$$

Example S.1.9. Solve

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 3, \quad t > 0, \quad \text{given that } u(0,t) = u(3,t) = 0,$$

$$u(x,0) = 5\sin(4\pi x) = 3\sin(8\pi x) + 2\sin(10\pi x), \quad |u(x,t)| < M \text{ (i.e., } u \text{ is bounded)}$$

More space for Example S.1.9.

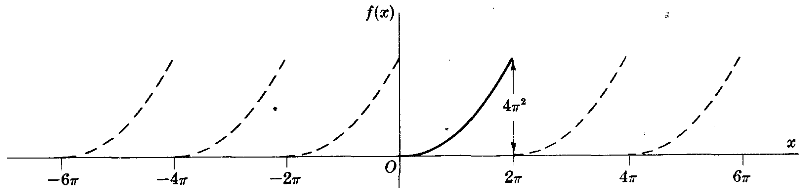
The previous example could be interpreted physically in the following way. A bar whose surface is insulated has a length of 3 units ($0 < x < 3$) and has a diffusivity of 2 units. If its ends are kept at temperature 0 and its initial temperature throughout is given by

$$u(x,0) = 5\sin(4\pi x) = 3\sin(8\pi x) + 2\sin(10\pi x),$$

, what is the temperature at position x at time t?

How does the solution for Example 0.1.9 change if the initial condition on $u(x,0)$ changes to a generic function $u(x,0) = f(x)$?

Section S.2
Fourier Series



Objectives:

- Analyze properties of Fourier series and use them for function expansions
- Use Fourier series to solve to PDEs

Recall from Section 5-5 that trigonometric integrals are useful in finding least squares approximations to functions in $C[-\pi,\pi]$ (or, as we could have said with some small alterations, $C[-L,L]$). In general, our functions are defined everywhere but are **periodic**, meaning that after a certain amount of space they repeat. For the sake of simplicity, we will assume that our period is centered at the origin so that we only need to focus on the interval $[-L,L]$ to create our Fourier coefficients.

Definition S.2.1. A function $f(x)$ has a **period** P or is **periodic** if for all x, $f(x+P)=f(x)$, where P is a positive constant. The least value of $P>0$ is called *the* **period** of $f(x)$.

The function $\sin(x)$ has periods of 2π, 4π, 6π, and $m\pi$ for m even, but we say its period is 2π. This period shortens if we consider $\sin(nx)$ for n a positive integer - the period of this latter function is $\frac{2\pi}{n}$.

We will extend our definition of Fourier coefficients slightly to accommodates for periods different from 2π:

Definition S.2.2. Let $f(x)$ be defined in the interval $[-L,L]$ and have period of $2L$. The **Fourier series** corresponding to $f(x)$ is defined to be

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right),$$

where the **Fourier coefficients** a_n and b_n are

$$a_n=\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$$

$$b_n=\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

Recall from the beginning of Section S-1 that it is an interesting question whether these Fourier series are even equal to the function $f(x)$ at all. They certainly are that way in $C[-\pi, \pi]$, but only because functions in this space satisfy what are known as the **Dirichlet conditions**:

Theorem S.1. *Suppose that*

(i) $f(x)$ *is defined and single-valued except possibly at a finite number of points in* $(-L, L)$

(ii) $f(x)$ *is periodic with period* $2L$

(iii) $f(x)$ *and* $f'(x)$ *are piecewise continuous in* $(-L, L)$.

Then the Fourier series of f *converges to*

(a) $f(x_0)$ *if* x_0 *is a point of continuity, or*

(b) $\dfrac{\lim_{x\to x_0+} f(x) + \lim_{x\to x_0-} f(x)}{2}$ *if* x_0 *is a point of discontinuity.*

Example S.2.1. Find the Fourier coefficients and Fourier series corresponding to the following function (with period 2π):

$$f(x) = \begin{cases} 0 & -5 < x < 0 \\ 3 & 0 < x < 5 \end{cases}$$

Recall that a function $f(x)$ is **odd** if $f(-x) = -f(x)$. Functions such as x^3 and $\sin(x)$ are odd functions.

Meanwhile, a function $f(x)$ is **even** if $f(-x) = f(x)$. Functions such as $2x^6 - 5$, $\cos(x)$, and $e^x + e^{-x}$ are even functions.

Whenever a function defined on $(-L, L)$ is even, then by the equations above we only need to define the function on $(0, L)$ - for the rest of the function (outside of $x = 0$), $f(-x) = f(x)$. We leave it to the reader to see that plugging an even function in for the Fourier coefficients b_n will always yield a value of zero.

Odd functions are also only typically defined on $(0, L)$, as they can be defined on $(-L, 0)$ using the equation $f(-x) = -f(x)$. We leave it to the reader to see that plugging an odd function in for the Fourier coefficients a_n will always yield a value of zero.

Hence we have the following definitions:

Definition S.2.3. The **half-range Fourier sine series** comes from using the following Fourier coefficients:

$$a_n = 0, \quad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx.$$

The **half-range Fourier cosine series** comes from using the following Fourier coefficients:

$$b_n = 0, \quad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx.$$

Example S.2.2. Expand $f(x) = \sin(x)$, $0 < x < \pi$, in a Fourier cosine series.

Example S.2.3. Expand $f(x) = x$, $0 < x < \pi$ in a half-range (a) sine series, and (b) cosine series.

Example S.2.4. Returning to Example S.1.9, find the temperature of the bar if the initial temperature $u(x, 0) = 25$.

Example S.2.5. A bar of length L, whose entire surface is insulated including its ends at $x = 0$ and $x = L$, has initial temperature $f(x)$. Determine the subsequent temperature of the bar.

More space for S.2.5

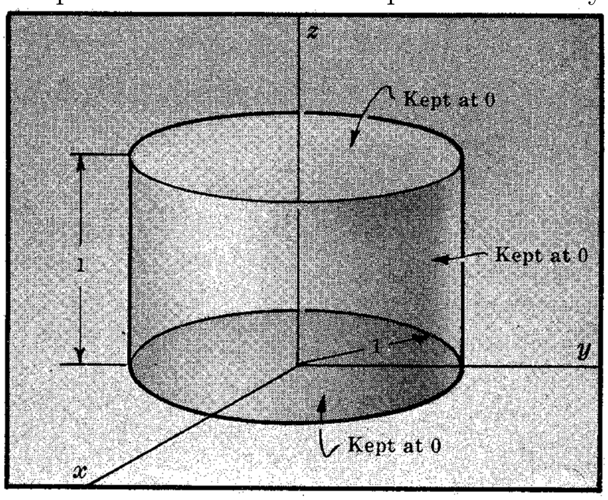
Section S.3
Bessel Functions

Objectives:

- Use the Method of Frobenius to calculate Bessel functions of the first kind
- Apply Bessel functions to applications with circular and cylindrical objects

We begin with an example:

Example S.3.1. A solid conducting cylinder of unit height and radius and with diffusivity κ is initially at temperature $f(r, z)$. (That is, the temperature does not rely on the angle θ.) The entire surface is suddenly lowered to temperature zero and kept at this temperature. Find the temperature at any point of the cylinder at any subsequent time.



One of the equations we paused on comes up frequently in physical situations involving circles and cylinders where the equation begin studied in includes the Laplacian. Indeed, we could apply a similar method of solution to Laplace's equation in cylindrical coordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

and get a similar answer. This particular equation has a solution called a *Bessel function of order zero.*

Let's take a look at the theory behind Bessel functions:

Definition S.3.1. **Bessel's differential equation** is a differential equation of the form

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

where $n \geq 0$ is a constant. The general solution of this equation is given by

$$y = c_1 J_n(x) + c_2 Y_n(x),$$

where $J_n(x)$ is a **Bessel function of the first kind**, which has a finite limit as $x \to 0$ and $Y_n(x)$ is a **Bessel function of the second kind**, which has infinite limit as $x \to 0$.

In this section we will only deal with properties that are bounded throughout - hence we will have no use for Bessel functions of the second kind.

One method of finding solutions to linear differential equations is called the **method of Frobenius**. With this method we assume the equation has a solution of the form

$$y = \sum_{k=-\infty}^{\infty} c_k x^{k+\beta},$$

very much like a power series that can have negative or fractional powers. We will assume that $c_k = 0$ for $k < 0$ so that this solution begins with a c_0-term.

Our strategy will be this: substitute this formula into the given PDE. Start by solving for β by finding an **indicial equation**. Then using this β, find the values of the constants c_k. Plugging all of those back in will give us our solution.

We will use a couple new tools in our solution below: we define the **Gamma function** $\Gamma(n) = \int_0^\infty x^{n-1}e^x \, dx$. In particular, if n is a positive integer, then $\Gamma(n+1) = (n)!$.

We will also see use of the **double factorial**, $n!!$, which is equal to $n(n-2)(n-4)\cdots 2$ if n is even or $n(n-2)(n-4)\cdots 1$ if n is odd.

Example S.3.2. Use the method of Frobenius to find series solution of Bessel's differential equation $x^2y'' + xy' + (x^2 - n^2)y = 0$.

Assume our solution of the form $y = \sum c_k x^{k+\beta}$ as given on the previous page. Then plugging in this solution, we get

$$\begin{aligned}(x^2 - n^2)y &= \sum c_k x^{k+\beta} + 2 - \sum n^2 c_k x^{k+\beta} = \sum c_{k-2} x^{k+\beta} - \sum n^2 c_k x^{k+\beta} \\ xy' &= \sum (k+\beta) c_k x^{k+\beta} \\ x^2 y'' &= \sum (k+\beta)(k+\beta-1) c_k x^{k+\beta}.\end{aligned}$$

We add these terms together to get that

$$\sum [(k+\beta)(k+\beta-1)c_k + (k+\beta)c_k + c_{k-2} - n^2 c_k] x^{k+\beta} = 0.$$

Now this function is *identically* zero, so *every* coefficient of this series must be zero. Hence we must have, after simplifying,

$$[(k+\beta)^2 - n^2]c_k + c_{k-2} = 0.$$

Let's analyze what happens at a particular value of k, like $k = 0$. Since $c_k = 0$ for $k < 0$, we get $c_{-2} = 0$, so we get an **indicial equation** $(\beta^2 - n^2)c_0 = 0$. Assuming that $c_0 \neq 0$ (otherwise we reset to another nonzero c-value and call that c_0), we get that $\beta^2 = n^2$.

Technically here, $\beta = \pm n$, yielding two cases. Both cases would need to be considered for a full treatment, but we will only consider the case where $\beta = n$ - the other case eventually develops into Bessel functions of the second kind. Our full equation then becomes (after simplifying)

$$k(2n + k)c_k + c_{k-2} = 0.$$

We observe what happens with this equation as we put in larger values for k:

$$c_1 = 0, \quad c_2 = -\frac{c_0}{2(2n+2)}, \quad c_2 = 0, \quad c_4 = -\frac{c_2}{4(2n+4)} = \frac{c_0}{2 \cdot 4(2n+2)(2n+4)}, \quad \ldots$$

Plugging in these constants into our method of Frobenius solution we get the series

$$\begin{aligned} y &= c_0 x^n + c_2 x^{n+2} + c_4 x^{n+4} + \cdots \\ &= c_0 x^n \left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2 \cdot 4(2n+2)(2n+4)} - \cdots \right]. \end{aligned}$$

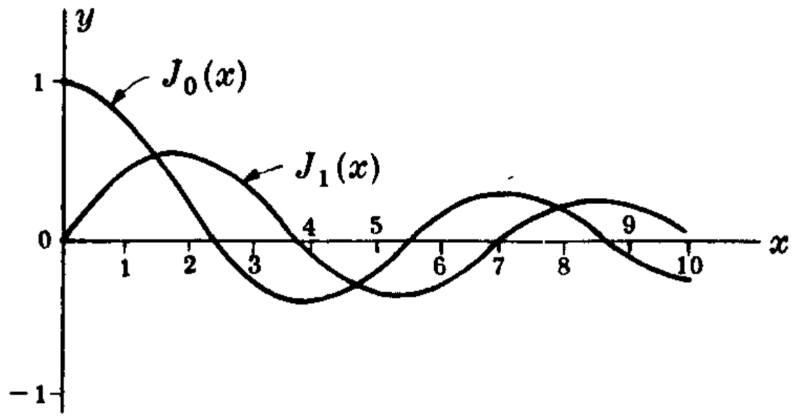
For many applications, we want a particular choice of c_0, which we set equal to $\frac{1}{2^n \Gamma(n+1)}$, where

$$\Gamma(n) = \int_0^\infty x^{n-1} e^x \, dx$$

is the **Gamma function**. Hence we have the function

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k} (2n)!!}{(2k)!!(2n+2k)!!},$$

where $n!! = n \cdot (n-2) \cdot (n-4)$ is the **double factorial** of n.



The Bessel functions of the first kind, while being complicated, should be thought of as sine and cosine functions. Observe the graphs of J_0 and J_1 above - notice any similarities?

An important distinction is that *the Bessel functions scale when the independent variable in Bessel's differential equation is scaled*. Let's see what this means:

Example S.3.3. Replace the variable x with the variable λu, where λ is some nonzero constant and u is otherwise independent. What does Bessel's differential equation become when written in terms of u?

Example S.3.4. A circular plate of unit radius has its plane faces insulated. If the initial temperature is $F(r)$ and if the rim is kept at temperature zero, find the temperature of the plate at any time.

In order to continue this example we need some formulas from *series* of Bessel functions to help us. Just as in this previous example, we will often run into a case where we have a superposition of solutions involving $J_n(\lambda_i x)$, where (λ_i) are the positive roots of $J_n(x) = 0$. In solving boundary conditions, we may end up with a line that looks like this:

$$f(x) = A_1 J_n(\lambda_1 x) + A_2 J_n(\lambda_2 x) + \cdots = \sum_{p=1}^{\infty} A_p J_n(\lambda_p x).$$

Just like the functions in a Fourier series, *Bessel functions of the first kind have orthogonality relations* in cases like these. These orthogonality relations allows us to compute what A_p must be:

$$A_p = \frac{2}{J_{n+1}^2(\lambda_p)} \int_0^1 x J_n(\lambda_p x) f(x)\, dx.$$

Instead of integrating this, we will leave our answers in this form. Using this information, complete the example from the previous page.

Example S.3.5. Finish Example 0.3.1. (If time is short, stop when considering boundary conditions.)

Example S.3.6. (If time allows) A drum consists of a stretched circular membrane of unit radius whose rim is fixed. If the membrance is struck so that its initial displacement is $F(r, \theta)$ and is then released, find the displacement at any time.