Proof Packet. This is worth 10% of your grade and will be updated as we proceed through the class. It is possible to receive more than 100% on this packet. Ways of receiving full points and extra credit on this packet include but are not limited to: answering every question in this first section correctly, presenting solutions to these problems in class, and making strong attempts at the problems in the "Challenging but Useful" section. You are encouraged to collaborate with students in this class on these problems - you may not use any source outside of the professor, your peers, or the course materials outlined in the syllabus for this class.

1. Produce an infinite collection of sets $A_1, A_2, A_3, \dots$ with the following properties:
   (a) Every A_i has an infinite number of elements.
   (b) If $i \neq j$, $A_i \cap A_j = \varnothing$.
   (c) $\bigcup_{i=1}^{\infty} A_i = \mathbb{N}$.
2. Show, using just the field axioms, that for every $n \in \mathbb{N}$ we have $n^2 \geq n$.
3. Show that the dyadic rationals $\{\frac{m}{2^n} : m \in \mathbb{Z}, n \in \mathbb{N}\}$ are dense in $\mathbb{R}$.
4. Prove that for all $k \in \mathbb{R}$, $\lim_{n\to\infty} \frac{k^n}{n!} = 0$.
5. (a) Show that if (x_n) converges in $\mathbb{R}$, then the sequence given by the **Cesàro mean**
   $$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$
   also converges to the same limit.
   (b) Give an example to show that it is possible for the sequences (y_n) of averages to converge even if (x_n) does not.
6. Using theorems about sequences, prove that for $s_n = \frac{1-n}{2^n}$, the sequence (s_n) converges to 0. (Hint: prove that $n < \left(\frac{3}{2}\right)^n$. Then use the Squeeze Theorem.)
7. Find a monotonic subsequence in the sequence (r_n), which consists of all rational numbers in the interval $(0, 1)$ arranged in some order.
8. Suppose (x_n) is a sequence such that every subsequence (x_{n_i}) has a further subsequence $(x_{n_{m_i}})$ that converges to a fixed $x \in \mathbb{R}$. Prove that $x_n \to x$.

9. Let $s_n = (-1)^{n+1}\frac{1}{n}$ be the **alternating harmonic series**. Show that $\sum s_n$ converges to a number between 1/2 and 1.

   (a) First show that $x_n := \sum_{k=1}^{n} s_k$ satisfies $\frac{1}{2} \leq x_n \leq 1$ for all n.

   (b) Complete the proof.

10. Show that, if $(I_\alpha)_{\alpha \in A}$ is a collection of disjoint open intervals, then the collection is countable. (Hint: $\mathbb{Q}$ is dense in $\mathbb{R}$.)

11. We give another definition for continuity of a function f.

    (a) Let (a, b) for $a < b$ be an open interval (here and elsewhere for similar sets, a or b can be $-\infty$ or ∞, respectively, if needed). Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that there is an open interval (c, d) such that $f((c, d)) \subset (a, b)$ whenever (a, b) has non-empty intersection with the range of f.

    (b) Now assume that, for all $a < b$, $f((c, d)) \subset (a, b)$ for some $c < d$. Prove that f is continuous.

12. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x + y) = f(x) + f(y)$. We say such a function f is **additive**. Further assume that f is continuous at $x = 0$.

    (a) Show that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

    (b) Show that $f(x) = kx$ for some $k \in \mathbb{R}$ whenever $x \in \mathbb{N}$. Then repeat for $x \in \mathbb{Z}$ and $x \in \mathbb{Q}$.

    (c) Show that $f(x) = kx$ on $\mathbb{R}$.

13. Show that there exists a uniformly continuous function on the set $[-1, 1]$ that is not Lipschitz continuous.

14. Since $\mathbb{Q}$ is countable, we can put all elements of $\mathbb{Q}$ in a sequence (this would be a surjective function from $\mathbb{N}$ into $\mathbb{Q}$). This is called an **enumeration of the rationals**, $(q_1, q_2, q_3, \dots)$. The elements of this sequence are not in any particular order and are certainly not in the "less than" order.

    Let $(q_1, q_2, q_3, \dots)$ be an enumeration of the rationals in $[0, 1]$. Consider the function $f : [0, 1] \to [0, 1]$ given by

$$f(x) = \sum_{q_n < x} \frac{1}{2^n}.$$

(This notation may be new: we are summing over all n such that $q_n < x$. You can also write the undersum expression as $\{n : q_n < x\}$.)

(a) Show that f is *strictly* increasing.

(b) Show that f is discontinuous at every rational number in $(0, 1)$. (Hint: prove that the left and right limits of f at a rational number are not equal.)

(c) Show that f is continuous at every irrational point in its domain.

15. Define the **Cantor set** C in the following way: Let $C_0 = [0, 1]$, and given C_k, let C_{k+1} be the set where each interval in C_k has the middle third taken out of it. For example, $C_1 = [0, 1/3] \cup [2/3, 1]$, $C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$, and so on. Let $C := \bigcap_{n=0}^{\infty} C_n$.

    (a) Prove that C is uncountable. (Hint: Cantor's Diagonalization might be useful.)

    (b) Prove that the complement of C is dense in $[0, 1]$.

16. A function f has a *symmetric derivative* at a point x if

$$f'_s(x) := \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$$

exists.

    (a) Show that $f'_s(x) = f'(x)$ at any point at which the latter exists.

    (b) Show that $f'_s(x)$ may exist even when f is not differentiable at x.

17. Find a function f that is differentiable at $x = 0$ but is not differentiable at any other point.

18. If f is twice differentiable on an open interval containing a and f'' is continuous at a, show

$$\lim_{h \to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = f''(a).$$

(Tools to consider using: the Mean Value Theorem, L'Hopital's Rule, and the symmetric derivative.)

19. For an interval $[a, b]$ where $a \geq 0$ and $n \in \mathbb{N}$, let $q_n := \sqrt[n]{\frac{b}{a}}$. Let $P_n := \{a, aq_n, aq_n^2, \ldots, aq_n^{n-1}, aq_n^n = b\}$. Calculate $\displaystyle\int_a^b \frac{1}{x^2}\,dx$ by calculating upper and lower sums with these partitions as n increases.
    (a) First give a formula for $U(f, P_n)$. (Don't forget that q is written in terms of n.)
    (b) Take a limit of this formula in n and simplify.
    (c) Prove $|U(f, P_n) - L(f, P_n)|$ limits to 0 as n increases.

20. We show that, for any integrable function f, $|f|$ is also integrable and that $\left|\int_a^b f\,dx\right| \leq \int_a^b |f|\,dx$.
    (a) First, let f be bounded on $[a, b]$. Define the following variables:

    $$M = \sup\{f(x) : x \in A\}, \quad m = \inf\{f(x) : x \in A\}$$
    $$M' = \sup\{|f(x)| : x \in A\}, \quad \text{and} \quad m' = \inf\{|f(x)| : x \in A\}.$$

    Show that $M - m \geq M' - m'$. (Hint: break into cases.)
    (b) Show that if f is integrable on the interval $[a, b]$, then $|f|$ is also integrable on this interval.
    (c) Noting that $f \leq |f|$ and $-f \leq |f|$, complete the proof using Problem 1.

Challenging but Useful. In what follows are some theorems we are equipped to prove. Step-by-step instructions are given. Any satisfactory work on these proofs is eligible for extra credit.

1. Do the following steps to prove the Schröder-Bernstein Theorem. This theorem states: "Let $f : X \to Y$ and $g : Y \to X$ be one-to-one functions on their respective sets. Then there exists a bijective function $h : X \to Y$ (and hence $|X| = |Y|$)."

   Let's set a destination point for our proof. Let $\sqcup$ denote the disjoint union of sets. We will partition $X = A \sqcup A'$ and $Y = B \sqcup B'$ in such a way that f maps A onto B and that g maps B' onto A'. This means $f|_A : A \to B$ is bijective, as is $g|_{B'} : B' \to A'$, so their inverses exist. We will then define $h : X \to Y$ to be

   $$h(x) = \begin{cases} f|_A(x) & x \in A \\ g|_{B'}^{-1}(x) & x \in B \end{cases}.$$

a) Show that h is well-defined (that is, it does not accidentally map one point to two points), then show it is bijective.

b) Set $A_1 = X \setminus g(Y) = \{x \in X : x \notin g(Y)\}$. (If $A_1 = \varnothing$, then $g(Y) = X$ and g is bijective.) Inductively define $A_{n+1} = g(f(A_n))$. Show that the collection $(A_n)_{n\in\mathbb{N}}$ consists of pairwise disjoint sets in X. Show that, similarly, $(f(A_n))_{n\in\mathbb{N}}$ consists of pairwise disjoint sets in Y.

c) Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} f(A_n)$. Show that f maps A onto B.

d) Let $A' = X \setminus A$ and $B' = Y \setminus B$. Show g maps B' onto A'.

2. Prove, *using the limit definition,* that for all $k \in \mathbb{N}$, $\lim_{n\to\infty} \frac{k^n}{n!} = 0$.

3. There is a separate proof of Bolzano-Weierstrass Theorem that does not require the fact that every sequence contains a monotone subsequence. We will walk through the steps here. We recall the theorem: *every bounded sequence* (x_n) *contains a convergent subsequence.*

a) Since (x_n) is bounded, the elements of this sequence are all contained in some interval $[-M, M]$ for some $M \in \mathbb{R}$. Show that there are infinitely many elements of (x_n) in either $[-M, 0]$ or $[0, M]$.

b) Let A^1 be the set found above such that there are infinitely many elements of (x_n) in A^1. Pick a_{n_1} to be an element in A^1; prove that there are infinitely many elements of x_n in A^1 such that $n > n_1$.

c) Let (x_{n_k}) be the subsequence of elements in A^1 such that $n > n_1$. Let's divide A^1 in two: let m_1 be the midpoint of A^1, let $A^1_- := \{x \in \mathbb{R} : x \leq m_1\} \cap A$, and let $A^1_+ := \{x \in \mathbb{R} : x \geq m_1\} \cap A$. Show that there are infinitely many elements of (x_{m_k}) in either A^1_- or A^1_+. (Do you see a pattern?)

d) Pick an element a_{n_2} to be an element of (x_{n_k}) in which of A^1_+ or A^1_- has infinitely many points. Prove that $n_2 > n_1$.

e) Give a general explanation of how to continue this process to find (a_{n_k}) and prove this is a subsequence of (x_n).

f) Complete the proof using the Squeeze Theorem.

4. Consider the collection $\mathcal{S}$ of Cauchy sequences in $\mathbb{Q}$. We define an equivalence relation: we say $(s_n) \sim (t_n)$ if the sequence $(s_n - t_n) \to 0$. This

equivalence relation induces a partition of $\mathcal{S}$ (remember from MATH 300?). We will call it $\mathcal{S}/\sim$. We define $\mathbb{R} = \mathcal{S}/\sim$.

We will now examine this definition to see if it holds up to scrutiny.

a) Prove that $\sim$ is indeed an equivalence relation.
b) Prove that $\mathbb{Q}$ is in $\mathcal{S}$.
c) Let $s := (a_n)$, $t := (b_n)$ be representatives of equivalence classes of $\mathcal{S}$. Define $s + t$ to be the equivalence class of $(a_n + b_n)$. Prove this definition is *well-defined*: that is, for *all* representatives of the equivalence classes of s and t, this definition always leads to the equivalence class of $(a_n + b_n)$.
d) Define $s \cdot t$ to be the equivalence class of the sequence $(a_n \cdot b_n)$. Prove this definition is well-defined as well.
e) Prove that, for any real number $s \neq 0$, there exists a real number t such that $s \cdot t = 1$.
f) Give a definition for a real number s to be *greater than* t, i.e., $s > t$.

5. In this problem we prove the **Contraction Mapping Theorem** for $\mathbb{R}$: Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that, for some constant c such that $0 < c < 1$,
$$|f(x) - f(y)| \leq c|x - y|$$
for all $x, y \in \mathbb{R}$. Then f has a **fixed point** y such that $f(y) = y$. Furthermore, given any $x \in \mathbb{R}$, the sequence $(x, f(x), f(f(x)), \dots)$ converges to y.

(a) First show that f is uniformly continuous on $\mathbb{R}$. (It is in fact **Lipschitz continuous**, which means $|f(x) - f(y)| \leq M|x - y|$ for some $M > 0$.)
(b) Pick a point $x_1 \in \mathbb{R}$. Now construct the sequence $(x_1, f(x_1), f(f(x_1)), \dots)$ so that in general, $x_{n+1} = f(x_n)$. Show that (x_n) is Cauchy.
(c) Let $y = \lim_n x_n$. Prove that $f(y) = y$. (Hint: use the definition of y as the limit of a sequence.)
(d) Prove that it is the *only* fixed point of f.
(e) Complete the proof - i.e., show that any other starting point for the sequence given in the theorem statement still converges to y.

6. We show the existence of a rather diabolical function.

a) Prove that the set $\{\frac{m}{\sqrt{p}} : p \text{ prime}, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. (You may use the fact that there are infinitely many prime numbers p, which is usually proven in a number theory course.)

b) Show that there exists a well-defined function f that is additive (i.e., $f(x+y) = f(x) + f(y)$) where f is not continuous at 0.

You might wish to use the following statement in proving your function is well-defined, and you may do so without proof: the set $\{\frac{1}{\sqrt{p}} : p \text{ prime}\}$ is linearly independent over $\mathbb{Q}$ (i.e., if $\sum_{i=1}^{n} \frac{q_i}{\sqrt{p_i}} = 0$ for $n \in \mathbb{N}$, $(q_i)_1^n \subset \mathbb{Q}$, and p_i distinct primes for all i, then $q_i = 0$ for all i).

c) Prove your function in (b) is in fact discontinuous everywhere.

7. We prove that the Darboux integral is equivalent to the Riemann integral on *continuous* functions. (It is in fact true for all functions in either class, but this makes our arguments easier.) Recall that a function f is **Riemann integrable** on $[a, b]$ if $\lim_{n\to\infty} \sum_{k=1}^{n} f(a + \frac{k}{n}\Delta x)\Delta x$ exists, where $\Delta x := \frac{b-a}{n}$. This summation is usually called the "right sum" in Calculus. Throughout this problem, let f be a *continuous* function on $[a, b]$.

(a) Prove that (Darboux) integrable functions are also Riemann integrable.

(i) Prove that the integrability criterion from class is equivalent to the **sequential integrability criterion**: for all $n \in \mathbb{N}$ there exist partitions P_n such that

$$U(f, P_n) - L(f, P_n) < \frac{1}{n}.$$

(ii) Let $P_n = \{a_1, \ldots, a_k\}$ be a partition of $[a, b]$. Find a Δx such that $\{a, a + \Delta x, a + 2\Delta x, \ldots, b\}$ is a refinement of P_n. (Hint: use induction on k, the number of elements in P_n.)

(iii) Complete the proof.

(b) Prove that a function is Riemann integrable (using right sums) if and only if $\lim_{n\to\infty} \sum_{k=1}^{n} f(x + \frac{k-1}{n}\Delta x)\Delta x$ exists (i.e., Riemann integrable using left sums) and that the limiting values in their definitions are the same.

(i) The function f is bounded on $[a, b]$ since it is continuous, so it is in fact uniformly continuous on $[a, b]$. Use the definition of uniform continuity to find an n such that $|x-y|<\frac{1}{n}$ implies $|f(x)-f(y)|<\frac{\varepsilon}{b-a}$.

(ii) Prove that it suffices to show that

$$\lim_{n\to\infty}\left|\sum_{k=1}^{n}\left(f(a+\frac{k}{n}\Delta x)-f(a+\frac{k-1}{n}\Delta x)\right)\Delta x\right|=0.$$

(iii) Using (i) and (ii), complete the proof. (Hint: triangle inequality.)

(c) Prove that Riemann integrable functions are also (Darboux) integrable and that their limiting values are the same.

(i) Let f be Riemann integrable. Consider $g:=\frac{f+|f|}{2}$ and $h:=\frac{f-|f|}{2}$. Prove that $f=g+h$ where g is Riemann integrable and nondecreasing, and h is Riemann integrable and nonincreasing.

(ii) We learn in our integral rules section that $\int_a^b f=\int_a^b g+\int_a^b h$. Prove using the sequential integrability criterion that the Darboux integral for g exists and agrees with the Riemann integral using right sums. Also prove that the Darboux integral for h exists and agrees with the Riemann integral using left sums.

(iii) Complete the proof.

(d) Suppose $f:[a,b]\to\mathbb{R}$ is continuous and $\int_a^x f=0$ for all *rational* $x\in[a,b]$. Show that $f(x)=0$ for all $x\in[a,b]$.