Advanced Calculus I

Texas A&M University

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$$\sqrt{2} := (1, 1.4, 1.41, 1.414, \dots)$$

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Big ideas:

- Proof and rigor

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Big ideas:

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Big ideas:

- Proof and rigor
- Ideas and creativity
- Beauty and overall cool stuff

Syllabus stuff

Take it away John

Review: Sets

A **set** A is a collection of objects. There are a few things that bar this from being a full definition, but we won't run into those here.

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DeMorgan’s Laws are useful to us, and we will prove the two-set version of them in our homework:

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We also have distributive laws: $B \cap (\bigcup_i A_i)^c = \bigcup_i (B \cap A_i)$, and $B \cup (\bigcap_i A_i)^c = \bigcap_i (B \cup A_i)$.

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A function can be **injective** (one-to-one), which means that for all $x \in B$, $f^{-1}(x) := f^{-1}(\{x\})$ has at most one element.

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Proving with Functions

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We can ask several things about functions. We can ask whether $f(x) \leq a$, $f(x) \geq a$, $f(x) = a$, what a domain/codomain needs to be for a function to be injective/surjective, what results from a **composition** $f \circ g$ of functions, constructing an **inverse function** f^{-1}, etc.

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Sometimes, to show that $f(x) = a$, we might elect to show $f(x) \leq a$ *and* $f(x) \geq a$. We might also show that $|f(x) - a| < \varepsilon$ for all $\varepsilon > 0$. (Although this may seem complicated now, this strategy will be indispensable for us in this course.)

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Sometimes, to show that $f(x) = a$, we might elect to show $f(x) \leq a$ *and* $f(x) \geq a$. We might also show that $|f(x) - a| < \varepsilon$ for all $\varepsilon > 0$. (Although this may seem complicated now, this strategy will be indispensable for us in this course.) If we are convinced "$f(x) = \infty$", we might instead claim $f(x) \geq M$ for all natural numbers M. (Another common strategy.)

Proof Overview

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Theorem

If n is odd, n^2 is odd.

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Theorem

If n is odd, n^2 is odd.

An Unconvincing Proof.

$1^2 = 1$. $3^2 = 9$. Even if you keep going through all the odd numbers, it's still odd. ■

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A Convincing Proof.

By definition a number n is odd if $n = 2k + 1$ for some $k \in \mathbb{Z}$. So let $n = 2k + 1$ for an arbitrary $k \in \mathbb{Z}$. Then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1$. So $n^2 = 2(2k^2 + 2k) + 1$, which is odd by definition. ■

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Theorem (Converse)

If n^2 is not odd, then n is not odd.

Theorem

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A Convincing Proof.

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Theorem (Contrapositive)

If n^2 is even, then n is even.

Theorem: $\sqrt{2}$ is irrational.

Proof:

Proof Idea:

Theorem: $\sqrt{2}$ is irrational.

Proof: Assume by way of contradiction (BWOC) that $\sqrt{2}$ is rational.

Proof Idea: We are setting up a *proof by contradiction*, which begins by assuming the opposite of the theorem's statement. Our goal is to arrive at an absurd statement.

Theorem: $\sqrt{2}$ is irrational.

Proof: Assume by way of contradiction (BWOC) that $\sqrt{2}$ is rational.

Then by definition of rational, $\sqrt{2} = \frac{p}{q}$, where p and q may be assumed to have no common factors.

Proof Idea: We are setting up a *proof by contradiction*, which begins by assuming the opposite of the theorem's statement. Our goal is to arrive at an absurd statement. We start by using the definition of rational to our advantage. If there were any common factors between p and q, we could divide them away.

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We apply our definitions in the hope of connecting back to something we know. We discover that p is even.

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What if we could also prove that q was even? Then both p and q have a factor of 2, which means that we can divide away the 2's. But this would be impossible: the 2's shouldn't have been there to begin with, since $\frac{p}{q}$ is simplified.

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Since $p = 2k$ **for some** $k \in \mathbb{Z}$**, we substitute to get** $(2k)^2 = 2q^2$**. Then** $4k^2 = 2q^2 \Rightarrow 2k^2 = q^2$**.**

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We don't have much to work with except the equation $p^2 = 2q^2$, so let's try playing with it some more, with the new knowledge that p is even.

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At this point we can say we went in the right direction: with just a few more steps we proved that q is even. It is hard to know when a contradiction proof is over, but once we clearly prove "p and q are both even" and explain why this is impossible, we will be done.

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But it is impossible for both p and q to be even (i.e., have a factor of 2) since p and q were assumed to have no common factors. **Our only assumption was that $\sqrt{2}$ was rational, so we must conclude that $\sqrt{2}$ is irrational.**

Success in Math 409

Tip

The people who succeed in this course are those who *create their own proof ideas* and are able to *translate their proof ideas into proofs*. *Memorizing proofs and proof ideas is useful, but it is not enough.*

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After this: practice! Some say mathematics is a great big puzzle. It takes a while to put all the pieces together. Use this as an opportunity to make connections with all the math we have done up to this point.

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I am always happy to entertain a question you have, as is your help session tutor. The best way to reach me is via email: `jweeks03@tamu.edu`. More information about this course can be found in the syllabus.

Number Systems

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- From our previous slide, we know there are irrational numbers as well. *How many more are there?*

Cardinality

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Definition

We say that a set A is **infinite** if it is not finite. An infinite set has **cardinality** (roughly, “counting size") $\aleph_0$ if it has the same cardinality as the natural numbers.

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Question

How does the cardinality of $\mathbb{Q}$ compare to the cardinality of $\mathbb{N}$?

Theorem: $|\mathbb{Q}| = |\mathbb{N}|$

Proof:

Proof Idea: It is a little difficult to think about comparing $\mathbb{N}$ and $\mathbb{Q}$. $\mathbb{Q}$ clearly contains $\mathbb{N}$ and seems to have a LOT more elements! However, we are going to try something unintuitive and prove that their “counting size" is the same.

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Let’s begin by stating what we want to do. With something this tricky, it’s good to let the audience know what’s up. Note that the claim follows from our goal by definition.

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A LOT of functions we can think of don't work: $f(n) = n$ is injective but not surjective. The trick is thinking about how to count *all numbers of the form* $\frac{m}{n}$ in a row.

Here's one way to try:

n	1	2	3	4	$\cdots$
1	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\cdots$
2	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\cdots$
3	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\cdots$
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n	1	2	3	4	$\cdots$
1	(1) $\frac{1}{1}$ $\longrightarrow$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\cdots$
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Given the table above, let a function h **assign each natural number sequentially in the order suggested by the arrows. (For example,** $h(1) = \frac{1}{1}$, $h(2) = \frac{2}{1}$, $h(3) = \frac{1}{2}$**, and so on.)**

Proof Idea: We want to be very clear about how we are assigning the values of this function. It's okay and even recommended to provide tables, graphs, and figures where useful.

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However, we have not yet mapped any numbers to the negative rationals. What is a way we can make this change?

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This maps $\mathbb{N}$ onto the positive rational numbers. Define $f : \mathbb{N} \to \mathbb{Q}$ to be $f(k) = (-1)^{k+1} \cdot h(\lceil \frac{k}{2} \rceil)$; then by construction this function maps onto $\mathbb{Q}$.

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We introduce the **ceiling function** $\lceil a \rceil$, which rounds a up to the nearest whole number. Notice that $\lceil \frac{k}{2} \rceil$ will output numbers in the following order: 1, 1, 2, 2, 3, 3,. . . .

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The $(-1)^{k+1}$ factor will output numbers like this: 1, -1, 1, -1, ... So the product of these two alternates between positive and negative numbers.

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Proof Idea:

There's one more issue though: we promised a bijection and were only able to give a surjection. (Note $f(1) = f(5)$, for example.)

Theorem: $|\mathbb{Q}| = |\mathbb{N}|$

Proof: We will define a particular function $f : \mathbb{N} \to \mathbb{Q}$ and prove that this function is a **surjection**.

n	1	2	3	4	$\cdots$
1	(1) $\frac{1}{1}$	$\frac{2}{1}$ (2)	$\frac{3}{1}$ (6)	$\frac{4}{1}$ (7)	$\cdots$
2	(3) $\frac{1}{2}$	$\frac{2}{2}$ (5)	$\frac{3}{2}$ (8)	$\frac{4}{2}$	$\cdots$
3	(4) $\frac{1}{3}$	$\frac{2}{3}$ (9)	$\frac{3}{3}$	$\frac{4}{3}$	$\cdots$
4	$\frac{1}{4}$ (10)	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\cdots$

Given the table above, let a function h assign each natural number sequentially in the order suggested by the arrows. (For example, $h(1) = \frac{1}{1}$, $h(2) = \frac{2}{1}$, $h(3) = \frac{1}{2}$, and so on.)

This maps $\mathbb{N}$ onto the positive rational numbers. Define $f : \mathbb{N} \to \mathbb{Q}$ to be $f(k) = (-1)^{k+1} \cdot h(\lceil \frac{k}{2} \rceil)$; then by construction this function maps onto $\mathbb{Q}$.

Proof Idea:

There's one more issue though: we promised a bijection and were only able to give a surjection. (Note $f(1) = f(5)$, for example.)

The reason we can be okay with this is due to the **Schröder-Bernstein Theorem**: if there is an injection and a surjection between two sets, then there is a bijection between them too.

—

Since $f(n) \to n$ is an injection from $\mathbb{N}$ to $\mathbb{Q}$, Schröder-Bernstein completes the proof.

Theorem: $|\mathbb{N}| < |\mathbb{R}|$

Proof:

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We will prove this one using proof by contradiction, since it seems near impossible to show the *lack* of a bijection directly. (Why is this an appropriate start to the proof?)

Theorem: $|\mathbb{N}| < |\mathbb{R}|$

Proof: Assume BWOC that f is a bijection between $\mathbb{N}$ and $[0, 1)$.

Write down the values of f in a table like this:

$$\begin{array}{c|cccccc} f(1) & 0 & . & a_1^1 & a_2^1 & a_3^1 & \cdots \\ f(2) & 0 & . & a_1^2 & a_2^2 & a_3^2 & \cdots \\ f(3) & 0 & . & a_1^3 & a_2^3 & a_3^3 & \cdots \\ \vdots & \vdots & . & \vdots & \vdots & \vdots & \ddots \end{array}$$

It suffices to show that there is a real number that is not present in this table of values.

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The table lines up each of the values by the decimal point. Each number past the decimal point is represented by a variable. (For example, $f(1) = a_1 * 0.1 + a_2 * 0.01 + \cdots$.)

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Create a number $k = 0.k_1 k_2 k_3 \ldots$ **such that** $k_i = 5$ **or** 6**, whichever** a_i^i **is not, for all** $i \in \mathbb{N}$**.**

Proof Idea:

To construct the real number we are looking for, we will make it different than the number $f(n)$ in the nth decimal place. We could be more general by just saying "let k_i be a different digit than a_i^i", as long as we avoid a few fringe cases.

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Then for any n, $k \neq f(n)$ since it is different from $f(n)$ in the nth decimal place.

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But since $k \in [0, 1)$ is a real number, this means k is not in the image of f. So f is not a bijection, yielding our contradiction.

A Few More Theorems

Definition

We say a set X with finite cardinality or with cardinality $\aleph_0$ is **countable**. If X is not countable, we say it is **uncountable**.

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Proof.

By definition of countable, for each A_i there is a surjective function $f_i : \mathbb{N} \to A_i$. (Why is f_i not necessarily bijective?) There are countably many sets in I, so there exists a surjective function $g : \mathbb{N} \to I$.

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$i \backslash k$	1	2	3	4	$\cdots$
1	$f_{g(1)}(1)$	$f_{g(1)}(2)$	$f_{g(1)}(3)$	$f_{g(1)}(4)$	$\cdots$
2	$f_{g(2)}(1)$	$f_{g(2)}(2)$	$f_{g(2)}(3)$	$f_{g(2)}(4)$	$\cdots$
3	$f_{g(3)}(1)$	$f_{g(3)}(2)$	$\ddots$	$\vdots$	$\cdots$
4	$f_{g(4)}(1)$	$f_{g(4)}(4)$	$\cdots$	$f_{g(i)}(k)$	$\cdots$

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$$\mathbb{R} = \bigcup_{n=-\infty}^{\infty} [n, n + 1).$$

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$$\mathbb{R} = \bigcup_{n=-\infty}^{\infty} [n, n+1).$$

Can you finish the proof? ∎

Axioms of $\mathbb{R}$

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The set $\mathbb{R}$ satisfies what are known as the *field axioms*, which roughly says that $\mathbb{R}$ plays nicely with addition and multiplication.
A1-A4 deal with addition, **M1-M4** deal with multiplication, and **AM1** deals with how the two operations interact with each other.

AM1 For any $a, b, c \in \mathbb{R}$ the identity $(a + b)c = ac + bc$ is true.

A1 For any $a, b \in \mathbb{R}$, $a + b \in \mathbb{R}$, and $a + b = b + a$.

A2 For any $a, b, c \in \mathbb{R}$, the identity

$$(a + b) + c = a + (b + c)$$

is true.

A3 There is a unique number $0 \in \mathbb{R}$ so that, for all $a \in \mathbb{R}$,

$$a + 0 = 0 + a = a.$$

A4 For any number $a \in \mathbb{R}$ there is a corresponding number denoted by $-a$ with the property that

$$a + (-a) = 0.$$

M1 For any $a, b \in \mathbb{R}$ there is a number $ab \in \mathbb{R}$ and $ab = ba$.

M2 For any $a, b, c \in \mathbb{R}$ the identity

$$(ab)c = a(bc)$$

is true.

M3 There is a unique number $1 \in \mathbb{R}$ so that

$$a1 = 1a = a$$

for all $a \in \mathbb{R}$.

M4 For any number $a \in \mathbb{R}$, $a \neq 0$, there is a corresponding number a^{-1} with the property that

$$aa^{-1} = 1.$$

Order of $\mathbb{R}$

$\mathbb{R}$ also satisfies some additional axioms that make it into an **ordered field**. Whereas a set satisfying **A1-A4**, **M1-M4**, and **AM1** can be called a **field**, any field satisfying **O1-O4** below can be called ordered:

O1 For any $a, b \in \mathbb{R}$, exactly one of the statements $a = b$, $a < b$, or $b < a$ is true.

O2 For any $a, b, c \in \mathbb{R}$, if $a < b$ and $b < c$, then $a < c$.

O3 For any $a, b \in \mathbb{R}$, if $a < b$, then $a + c < b + c$ for any $c \in \mathbb{R}$.

O4 For any $a, b \in \mathbb{R}$, if $a < b$ then $ac < bc$ for any $c \in \mathbb{R}$. . . .

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O4 For any $a, b \in \mathbb{R}$, if $a < b$ then $ac < bc$ for any $c \in \mathbb{R}$ where $c > 0$.

Exercise: prove the arithmetic-geometric mean inequality using only the axioms we have discussed so far.

$$\sqrt{ab} \leq \frac{a+b}{2} \quad \text{where} \quad a, b > 0.$$

Theorem: $\sqrt{ab} \leq \frac{a+b}{2}$ for $a, b > 0$.

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Proof Idea: Let's start by working backwards. What's the most basic thing we need to work with here?

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We can start by multiplying by 2 and squaring both sides to get $4ab \leq (a+b)^2$.

Then after multiplying and combining like terms we get $2ab \leq a^2 + b^2$.

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It seems easiest to just prove the square of *any* number is non-negative.

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By O1, either $x = 0$, $x > 0$, or $x < 0$. If $x = 0$, then $0^2 = 0$ (why?), so $x^2 = 0$.

Proof Idea:

As simple as the idea seems, just the beginning of the proof illuminates that working from axioms can be a little tedious. *Everything* we say comes from an axiom - we can't use the things we have taken for granted before here.

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If $x > 0$**, then by O4,** $n \cdot n > n \cdot 0 \overset{M4}{=} 0$**. So** $n^2 > 0$**.**

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Hence $(a - b)^2 \geq 0$**. By applying AM1 twice, we get** $a^2 - 2ab + b^2 \geq 0$**.**

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We can now move on, since we can substitute any real number in for x, like $a - b$. (Can you explain each instance of AM1?)

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We can add $4ab$ to both sides by O3 to get $4ab \leq a^2 + 2ab + b^2$. Again, by applying AM1 twice, we have $4ab \leq (a + b)^2$.

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We can now move on, since we can substitute any real number in for x, like $a - b$. (Can you explain each instance of AM1?)

Let's take our backwards work from before and move forwards with it now. When we were combining like terms, we eventually got to $4ab \leq a^2 + 2ab + b^2$, so let's add $4ab$ to both sides.

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To be continued...

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The only thing left is to take square roots of both sides then apply O4... but we haven't proven that taking square roots preserves the order yet. This will be part of a homework assignment.

Bounds, Max, and Min

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The **closed interval** $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$ has a maximum b and a minimum a. The **open interval** $(a, b) := \{x \in \mathbb{R} : a < x < b\}$ has neither a maximum nor a minimum. However, both sets are considered **bounded** since they have both upper and lower bounds:

Definition

Let $E \subset \mathbb{R}$. M is an **upper bound** for E if $x \leq M$ for all $x \in E$. The number m is a **lower bound** for E if $x \geq m$ for all $x \in E$.

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Question: Is $\mathbb{N}$ bounded in $\mathbb{R}$?

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Question: Is $\mathbb{N}$ bounded in $\mathbb{R}$? **No**; it has a lower bound but no upper bound.

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Let $E \subset \mathbb{R}$. Then M is a **maximum** of E if $M \in E$ *and* M is an upper bound for E. Also, m is a **minimum** of E if $m \in E$ *and* m is a lower bound for E.

Bounds, Max, and Min

The **closed interval** $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$ has a maximum b and a minimum a. The **open interval** $(a, b) := \{x \in \mathbb{R} : a < x < b\}$ has neither a maximum nor a minimum. However, both sets are considered **bounded** since they have both upper and lower bounds:

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Question: Is $\mathbb{N}$ bounded in $\mathbb{R}$? **No**; it has a lower bound but no upper bound.

Sup, Inf, and Completeness

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The closed interval $[0, 1]$ has many lower bounds and upper bounds. The greatest of all lower bounds for the set is 0, and the least of all upper bounds for the set is 1.

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Let $E \subset \mathbb{R}$ be **bounded above** and nonempty.
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If E is unbounded above, we will say $\sup E = \infty$. Similarly, if E is unbounded below, we will say $\inf E = -\infty$.

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Answer: It depends on which ordered field you're using. This *is not true* if our ordered field is $\mathbb{Q}$. This is an *axiom* if our ordered field is $\mathbb{R}$. We say an ordered field where every set bounded above has a supremum is **complete**.

Proof that $\mathbb{Q}$ is Not Complete.

Consider the set $E = \{x \in \mathbb{Q} : x^2 < 2\}$.

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Proof that $\mathbb{Q}$ is Not Complete.

Consider the set $E = \{x \in \mathbb{Q} : x^2 < 2\}$. Then if $q \in \mathbb{Q}$ is an upper bound, let $p := q - \frac{q^2-2}{q+2}$. Note p is the subtraction of rational numbers, so p is rational. Since q is an upper bound, $q^2 - 2 > 0$ & $p < q$.

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Completeness Axiom of $\mathbb{R}$ A nonempty set of real numbers that is bounded above has a least upper bound.

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What relationships are there between $\sup A$, $\sup B$, $\inf A$, and $\inf B$?

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By a symmetric argument (since $A = -B$), $-\sup B = \inf A$.

Archimedean Property

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Theorem (Archimedean Property of $\mathbb{R}$)

The set of natural numbers $\mathbb{N}$ has no upper bound.

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Texas A&M University

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Let us see some consequences of this property before proving it:

Corollary (1)

Given any positive number y, no matter how large, and any positive number x, no matter how small, there exists an $n \in \mathbb{N}$ such that $nx > y$.

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Corollary (1)

Given any positive number y, no matter how large, and any positive number x, no matter how small, there exists an $n \in \mathbb{N}$ such that $nx > y$.

Corollary (2)

Given any positive number x, no matter how small, one can find a number $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.

Theorem: $\mathbb{N}$ has no upper bound.

Proof: Assume BWOC that $\mathbb{N}$ does have an upper bound.

Proof Idea: It is tricky to prove non-existence directly, so let's proceed by contradiction.

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This is where we use the completeness axiom. Although we cannot make any progress with this proof on grounds of merely having an upper bound, we can still disprove by showing there is no *least* upper bound. If there were any upper bound, then the infimum of those upper bounds would be a least upper bound for $\mathbb{N}$. (Prove this!)

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We restate what being a supremum means.

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Let $m \in \mathbb{N}$ be some natural number such that $m > x - 1$.

Proof Idea:

Notice that we can pick this number m because of what we just stated. Since $n \leq x - 1$ is not true for all natural numbers, there *exists* m such that the opposite is true.

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Then $m + 1 > x$. Note that $m + 1$ is also a natural number.

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Why did we add 1 to both sides of our inequality?

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But x is $\sup \mathbb{N}$, so $x \geq m + 1$. This is a contradiction.
The only thing we assumed was that $\mathbb{N}$ has an upper bound, so this must not be the case.

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It is a kind gesture to remind the audience what the contradiction leads up to.

$\mathbb{Q}$ is Dense in $\mathbb{R}$

The nature of $\mathbb{Q}$ in $\mathbb{R}$ is of great interest to us: even though $|\mathbb{Q}| < |\mathbb{R}|$, it seems as though it is evenly spaced throughout $\mathbb{R}$. In fact, it turns out that *every interval of* $\mathbb{R}$ *contains infinitely many points of* $\mathbb{Q}$. For $\mathbb{Q}$ being so relatively small, this comes pretty rarely; $\mathbb{N}$ is the same size of $\mathbb{Q}$, but there are plenty of intervals of $\mathbb{R}$ with *no* points of $\mathbb{N}$.

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Definition

A set $E \subset \mathbb{R}$ is **dense** in $\mathbb{R}$ if every interval (a, b) contains a point of E.

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Exercise: Prove that this definition is equivalent to the following definition for dense:

Definition

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Keep in mind that equivalence is an “if and only if".

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WRONG Proof!

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The issue is that x might not be rational! We can adapt this proof, but we will need to be a bit more careful than just picking $x + \frac{1}{n}$.

Theorem: $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof: Let $x < y$ and consider the interval (x, y). Our goal is to find a rational number in this interval.

By the Archimedean Property, there is a natural number $\frac{1}{n} < y - x$. Then
$ny > nx + 1$**.**

Proof Idea: We begin again just like we did in our incorrect proof. The idea is that we got the correct denominator of n. The length of the interval (x, y) exceeds $\frac{1}{n}$, so intuitively we must be able to find some $\frac{m}{n}$-type rational in (x, y).

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Here we use the following fact: for any real number x, there exists a natural number m such that $m \leq x < m + 1$. We will prove this in a homework exercise.

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By the former inequality,

$$m \leq nx + 1 < ny.$$

Dividing through by n, $\frac{m}{n} \leq x + \frac{1}{n} < y$.

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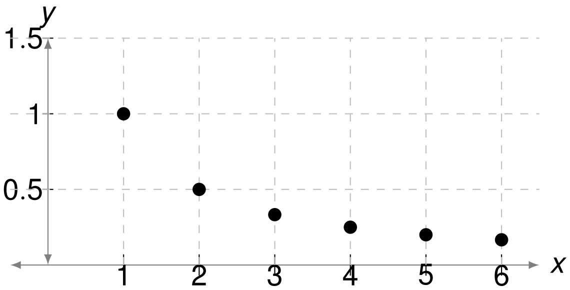
Here we use the following fact: for any real number x, there exists a natural number m such that $m \leq x < m + 1$. We will prove this in a homework exercise.

Some algebraic manipulation helps us complete the proof.

A common phrase in MATH 409 is “for all epsilon greater than 0, there exists a delta greater than 0". But what does it mean?



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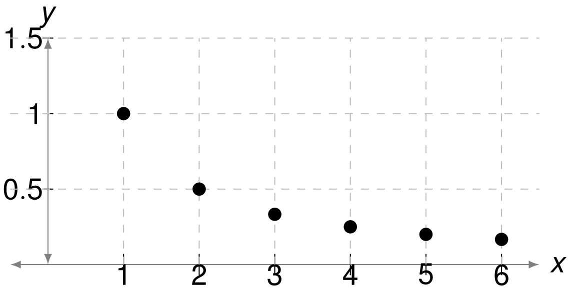
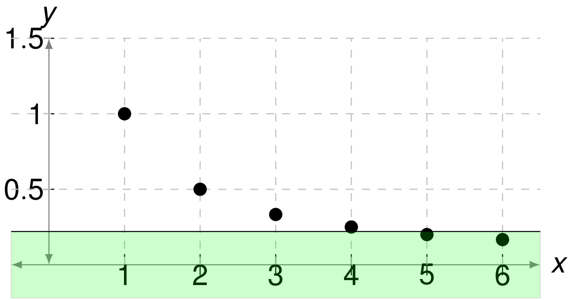
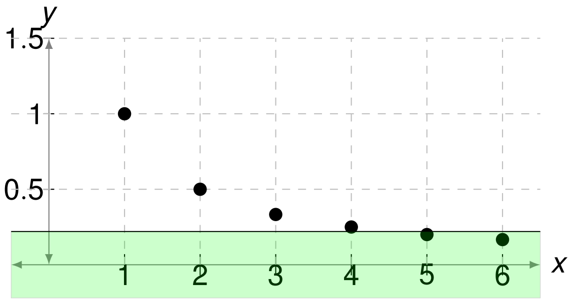


This graph denotes the sequence $a_n = \frac{1}{n}$ for $n \in \mathbb{N}$. We say (a_n) **converges** to 0 and write $a_n \to 0$. In Calculus II we could prove this using the **Monotone Convergence Theorem.**



We can start by understanding ε. ε is a window in our sequence or function values around a specified value. Here that window is centered around 0.



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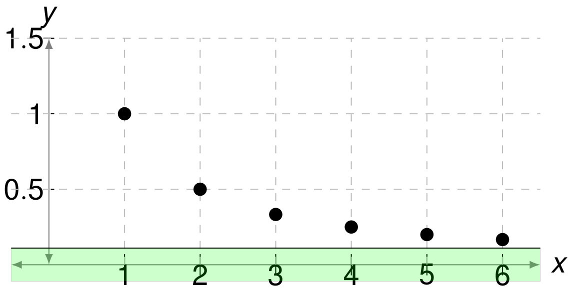

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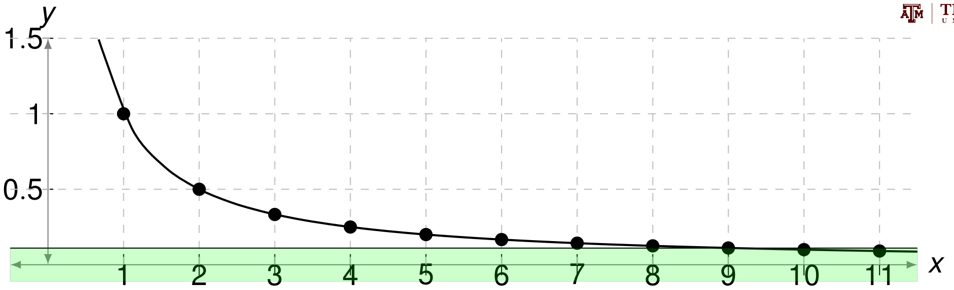
The “for all" quantifier is important! This sequence needs to eventually enter *any* window around 0. As our window shortens, the sequence is likely to get farther along before residing in the neighborhood. https://www.desmos.com/calculator/yfjleatok5



We can start by understanding ε. ε is a window in our sequence or function values around a specified value. Here that window is centered around 0.

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Theorem: $\lim_{n\to\infty} \frac{1}{n} = 0$.

Proof: Let $\varepsilon > 0$ be arbitrary.

Proof Idea: Since our ε-window must be arbitrarily small in order for the limit definition to work, we don't start by setting a value for ε but let it vary above 0.

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By definition of a sequential limit, then, $\lim_{n\to\infty} \frac{1}{n} = 0$.

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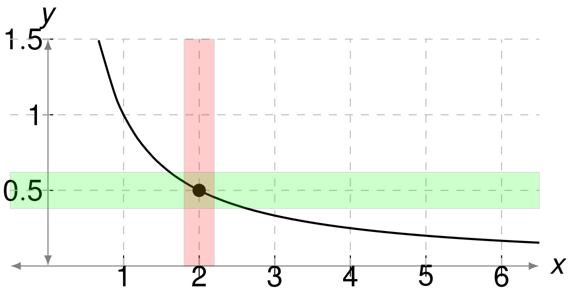
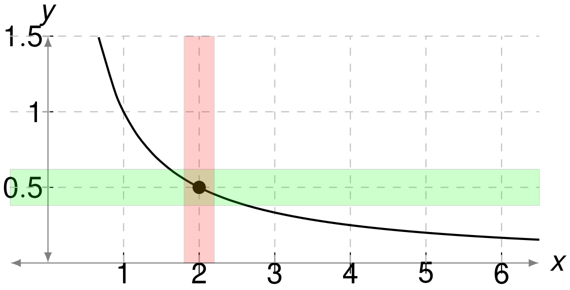


For functions (Chapter 5) we have a similar picture, but an x-window is introduced. The limit definition for a function uses the same ϵ-neighborhood as a sequential limit, but since we are not approaching infinity we need a δ-neighborhood as well.



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Observe, for the x-window above, for any x in this neighborhood, the function values $f(x)$ are in the y-window. This means $\lim_{x\to 2} \frac{1}{x} = \frac{1}{2}$.

Sequences are Everywhere

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Definition

A **sequence** is an ordered list. For mathematicians, this can be list of numbers, sets, functions, or even other sequences. If we do not specify, our sequences are *always infinite*.

Example

The sequence of even integers is $(2,4,6,8,\ldots)$.

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A function can define a sequence. For example, in our prelude we defined $a_n = f(n)$ where $f(x) = \frac{1}{x}$. Sequences of real numbers can be thought of as functions themselves with a domain of $\mathbb{N}$.

Theorem: The sequence $x_1 = \sqrt{2}$, $x_n = \sqrt{2 + x_{n-1}}$ converges to 2.

Proof: We will use the Monotone Convergence Theorem to prove this.

Proof Idea: This is a good chance to reviewing the notion of proof by induction. Remember: MCT says any bounded increasing (or decreasing) sequence converges.

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Remember that induction has two steps: prove for a base case (here x_1), and then prove that if something holds for x_k, then it holds for x_{k+1}. You are lining up the elements like dominos - the base case knocks down the first.

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Now note that $x_1 = \sqrt{2} < \sqrt{2 + \sqrt{2}} = x_2$.

Assume that $x_{k-1} < x_k$.

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We are now beginning another induction to show that $x_{n+1} > x_n$ for all n. (This is the definition of an increasing sequence.)

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What we have done tells us we can take a limit of both sides of our original equation.
We are about to square both sides and claim the limits still equal - we will prove this later.

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If $L := \lim_{n\to\infty} x_n$, then in squaring both sides we get $L^2 = \lim_{n\to\infty}(2 + x_{n-1}) = 2 + L$.

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Solving for L gets us $L = -1, 2$, and since all of our terms are positive we get that $L = 2$, completing the proof.

Convergence

In order to prove the Monotone Convergence Theorem, we need a *formal definition* of a limit. This is the purpose of MATH 409: **formalizing intuitive notions in order to prove strong theorems.**

Definition

Let (s_n) be a sequence of real numbers. We say that (s_n) **converges** to a number L and write $\lim_{n\to\infty} s_n = L$ provided that, for every $\varepsilon > 0$, there exists an integer $N \in \mathbb{N}$ so that, whenever $n \geq N$,

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Compare this to our $\frac{1}{n}$ example from our prelude, where $L = 0$.

Theorem: $\lim_{n\to\infty} \frac{n^2}{2n^2+1} = \frac{1}{2}$.

Proof: Fix an arbitrary $\varepsilon > 0$.

Proof Idea: The definition is asking us to find an N such that

$$\left| \frac{n^2}{2n^2+1} - \frac{1}{2} \right| < \varepsilon.$$

for $n \geq N$. It's much easier to work backwards.

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By getting a common denominator, we get this is equivalent to

$$\frac{1}{2(2n^2+1)} < \varepsilon.$$

(The absolute value disappeared since our inside number is already positive.)

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We are trying to solve for n, so we keep moving things around: we get $4n^2 + 2 > \frac{1}{\varepsilon}$, or

$$n^2 > \frac{1}{4}(\frac{1}{\varepsilon} - 2).$$

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Finally, recall that we get to *choose* the N we would like as long as it satisfies this inequality. This means N could be as large as we like, so it's okay to add $\frac{1}{2}$ to the right-hand side. Solving for n, $n > \frac{1}{2\sqrt{\varepsilon}}$.

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Then for $n \geq N$**,**

$$\left|\frac{n^2}{2n^2+1} - \frac{1}{2}\right| = \frac{1}{2(2n^2+1)} \leq \frac{1}{2(2N^2+1)}$$

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We now just walk through the rest of the proof. For an arbitrary $\varepsilon > 0$, we have picked an $N \in \mathbb{N}$; we only need show that this N fits the definition $|s_n - L| < \varepsilon$.

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By our bound, this is less than or equal to

$$\frac{1}{2}\frac{1}{1+\frac{1}{2\varepsilon}} \leq \frac{1}{2}2\varepsilon = \varepsilon.$$

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Although we can find many other limits much like we did in this proof, the real use of this definition is to build bigger theorems. You likely have other ways to find this limit - in this unit we will go through and prove more machinery to make finding limits simpler.

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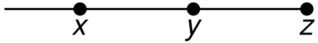
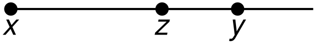
Fix $\varepsilon > 0$. Then by the above there exists an N such that $L - \varepsilon < s_N$. Since (s_n) is non-decreasing, $s_n \geq s_N$ for $n \geq N$, so $L - \varepsilon < s_n \leq L < L + \varepsilon$ for all $n > N$. Hence $|s_n - L| < \varepsilon$ as desired. ■

Uniqueness of Limits

The definition of a limit measures the smaller and smaller distance between points of a sequence and the limit those points converge to. Say we are comparing the distance between three points x, y, and z like below:





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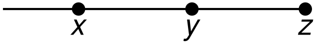
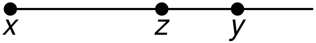


$$|x - y| < |x - z| + |z - y|$$



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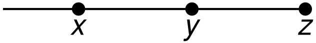

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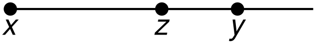

$$|x - y| = |x - z| + |z - y|$$

It is remarkably useful to combine these two possibilities into one inequality. This is known as the **Triangle Inequality**: for any $x, y, z \in \mathbb{R}$,

$$|x - y| \leq |x - z| + |z - y|.$$

Theorem (Uniqueness of Limits)

Suppose that $\lim_{n\to\infty} s_n = L_1$ *and* $\lim_{n\to\infty} s_n = L_2$ *are both true. Then* $L_1 = L_2$*.*

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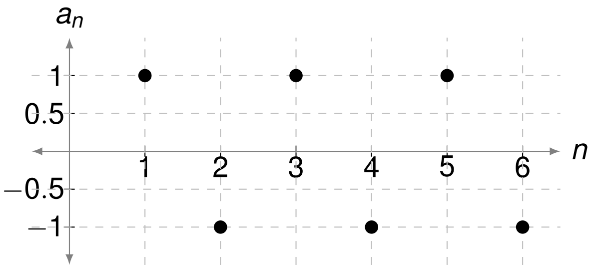
Before beginning, let's highlight that it's not as simple as substitution to solve this problem. What if there are multiple numbers that satisfy the definition of a limit?



$a_n = (-1)^{n+1}$

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Let $\varepsilon > 0$ be arbitrary. Then there exists an $N_1 \in \mathbb{N}$ such that $|s_n - L_1| < \varepsilon$ for $n \geq N_1$. There also exists an $N_2 \in \mathbb{N}$ such that $|s_n - L_2| < \varepsilon$ for $n \geq N_2$.

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$$|L_1 - L_2| =$$

∎

Theorem (Uniqueness of Limits)

Suppose that $\lim_{n\to\infty} s_n = L_1$ *and* $\lim_{n\to\infty} s_n = L_2$ *are both true. Then* $L_1 = L_2$.

Proof.

Let $\varepsilon > 0$ be arbitrary. Then there exists an $N_1 \in \mathbb{N}$ such that $|s_n - L_1| < \varepsilon$ for $n \geq N_1$. There also exists an $N_2 \in \mathbb{N}$ such that $|s_n - L_2| < \varepsilon$ for $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Then

$$|L_1 - L_2| = |(L_1 - s_N) + (s_N - L_2)| \leq |s_N - L_1| + |s_N - L_2| = 2\varepsilon.$$

The inequality above is due to the Triangle Inequality.

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The inequality above is due to the Triangle Inequality. Letting $\varepsilon \to 0$, we get that

$$|L_1 - L_2| = 0$$

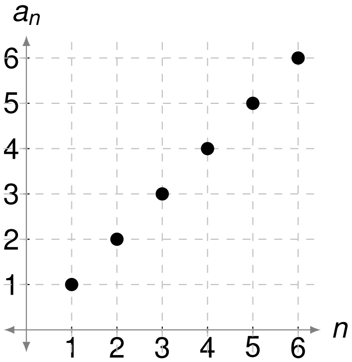
since it is smaller than any positive number. So $L_1 - L_2 = 0 \Rightarrow L_1 = L_2$. ∎

Divergence to Infinity

As we saw in our previous slide, not all sequences converge! Some have differing **subsequential limits** - some elements may tend toward one limit and others tend elsewhere. We will discuss these more following our discussion of subsequences.

Divergence to Infinity

As we saw in our previous slide, not all sequences converge! Some have differing **subsequential limits** - some elements may tend toward one limit and others tend elsewhere. We will discuss these more following our discussion of subsequences. Another way for sequences to diverge is also easy to see:



$a_n = n$

Theorem: $a_n = n$ **diverges to infinity.**

Proof:

Proof Idea:

Definition

A sequence (s_n) **diverges to infinity** if, for all $M \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that $s_n \geq M$ for all $n \geq N$.

Theorem: $a_n = n$ diverges to infinity.
Proof: Let $M \in \mathbb{R}$**.**

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We want to find an N such that $a_n \geq M$ for all $n \geq M$. Normally this takes some working backwards to find, much like when we were finding where a function converges.

Theorem: $a_n = n$ diverges to infinity.
Proof: Let $M \in \mathbb{R}$.
By the Archimedean Property, there exists a natural number N such that $N \geq M$. Hence for $n \geq N$, $a_n = n \geq N \geq M$.

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The Archimedean Property comes in handy once again! Remember that, if this N would not be possible to find, then M would be an upper bound for $\mathbb{N}$.

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By the Archimedean Property, there exists a natural number N such that $N \geq M$.

Hence for $n \geq N$, $a_n = n \geq N \geq M$.

So (a_n) diverges to infinity by definition.

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Theorem: $a_n = \dfrac{n^2+1}{n+1} \to \infty$.

Proof: Let $M \in \mathbb{N}$**.**

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Doing some long division, we get $\frac{n^2+1}{n+1} = n + \frac{1}{n+1} - \frac{n}{n+1} \geq M$.

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So it is acceptable to choose any n such that $n + \frac{1}{n+1} \geq M + 1$. (Why?)

Theorem: $a_n = \dfrac{n^2+1}{n+1} \to \infty$.

Proof: Let $M \in \mathbb{N}$.

Let $N := M + 1$. **Then** $a_N = \frac{M^2+2M+2}{M+1}$.

Proof Idea: Let's practice working backwards with this proof. We want to find an N such that, for any $n \geq N$, $\frac{n^2+1}{n+1} \geq M$.

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Note that we used that M is a natural number to make these definitions.

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After long division, we get
$a_N = M + 1 + \frac{1}{M+1} \geq M$.
Note that a_n is increasing. So for $n \geq N$, $a_n \geq a_N \geq M$. So $a_n \to \infty$.

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$\frac{n^2+1}{n+1} = n + \frac{1}{n+1} - \frac{n}{n+1} \geq M$.

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Note that we used that M is a natural number to make these definitions.

No need to prove that a_n is increasing... but how would you prove it?

A Quick Note: Absolute Value

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Knowing our way around absolute value becomes essential around these proofs. Let us gather what we know about absolute value here for reference. Assume $x, y, z \in \mathbb{R}$.

$$
\begin{aligned}
|x| &= 0 \text{ iff } x = 0 \\
|x-y| &= |y-x| \\
|x-y| &\le |x-z| + |z-y| && \leftarrow \quad \text{Triangle Inequality} \\
|x-z| &\ge ||x-y| - |y-z|| && \leftarrow \quad \text{Reverse Triangle Inequality} \\
|xy| &= |x||y| && \leftarrow \quad \text{Multiplicative}
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The first three rules (together with the fact that absolute value is never infinite and never negative) makes $|\cdot|$ a **norm** on $\mathbb{R}$.

Boundedness of Sequences

Definition

We say a sequence (s_n) is **bounded** if its range (collection of values) is a bounded set. I.e., there exists an $M \in \mathbb{R}$ such that

$$|s_n| \leq M$$

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Question: Are convergent sequences bounded? **Yes!** Proof upcoming.
Question: Are bounded sequences convergent?

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We say a sequence (s_n) is **bounded** if its range (collection of values) is a bounded set. I.e., there exists an $M \in \mathbb{R}$ such that

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Question: Are convergent sequences bounded? **Yes!** Proof upcoming.
Question: Are bounded sequences convergent? **No!** Consider $a_n = (-1)^{n+1}$.

Theorem

Every convergent sequence is bounded.

Proof.

Let $(s_n) \to L$.

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Let $(s_n) \to L$. Then there exists some N such that, for $n \geq N$, $|s_n - L| < 1$.

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Proof.

Let $(s_n) \to L$. Then there exists some N such that, for $n \geq N$, $|s_n - L| < 1$. So for $n \geq N$, $|s_n| \leq |s_n - L| + |L| < |L| + 1$.

Theorem

Every convergent sequence is bounded.

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Let $(s_n) \to L$. Then there exists some N such that, for $n \geq N$, $|s_n - L| < 1$. So for $n \geq N$, $|s_n| \leq |s_n - L| + |L| < |L| + 1$.
Let $M := \max\{|s_1|, |s_2|, \ldots, |s_{N-1}|, |L| + 1\}$.

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Every convergent sequence is bounded.

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Let $(s_n) \to L$. Then there exists some N such that, for $n \geq N$, $|s_n - L| < 1$. So for $n \geq N$, $|s_n| \leq |s_n - L| + |L| < |L| + 1$.
Let $M := \max\{|s_1|, |s_2|, \ldots, |s_{N-1}|, |L| + 1\}$.Then $s_n \in [-M, M]$ for all $n \in \mathbb{N}$, and (s_n) is bounded. ∎

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Corollary (Converse)

Theorem

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Corollary (Converse)

Every unbounded sequence diverges.

Theorem: $s_n = \sum_{k=1}^{n} \frac{1}{n}$ diverges.

Proof:

Proof Idea: Let's begin by understanding the sequence. $s_1 = 1$, $s_2 = 1 + \frac{1}{2}$, $s_3 = 1 + \frac{1}{2} + \frac{1}{3}$... This is a sequence of partial sums, whose limit is a **series**.

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Theorem: $s_n = \sum_{k=1}^{n} \frac{1}{n}$ diverges.
Proof: We will show that (s_n) is unbounded; the corollary on the previous slide will complete the proof.

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—

Since $s_{2^n} \geq 1 + \frac{n}{2}$, (s_n) is unbounded. This completes the proof.

Limit Laws

Let's return to a previous limit we took: $\lim_{n\to\infty} \frac{n^2}{2n^2+1} = \frac{1}{2}$. We can certainly prove this using the limit definition, but on first glance there seems to be a more intuitive approach:

$$\lim_{n\to\infty} \frac{n^2}{2n^2+1}$$

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You can see how many intermediate steps we had to take - let's go ahead and prove as many as we can with the tools we've developed so we can use them right away.

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Theorem (Sums/Differences of Limits)

Suppose $s_n \to s$, $t_n \to t$. *Then* $\lim_{n\to\infty}(s_n + t_n) =$ *and* $\lim_{n\to\infty}(s_n - t_n) =$.

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Suppose $s_n \to s$, $t_n \to t$. *Then* $\lim_{n\to\infty}(s_n + t_n) = s + t$ *and* $\lim_{n\to\infty}(s_n - t_n) = s - t$.

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Fix $\varepsilon > 0$. Choose N_1 such that $|s_n - s| < \frac{\varepsilon}{2}$ and N_2 such that $|t_n - t| < \frac{\varepsilon}{2}$. Let $N = \max\{N_1, N_2\}$. Then for $n \geq N$, $|(s_n + t_n) - (s + t)| \leq |s_n - s| + |t_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

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Theorem (Comparison Test I)

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Let $L := \lim_{n\to\infty} s_n$. Fix an $\varepsilon > 0$, and choose N such that $|s_n - L| < \varepsilon$. Note that

$$0 \leq s_n = (s_n - L) + L < L + \varepsilon.$$

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Proof.

Since $s_n \leq t_n$, $(t_n - s_n) \geq 0$. So $\lim_{n\to\infty} t_n - s_n \geq 0$ by the above theorem. By Sums of Limits Theorem, $\lim_{n\to\infty} s_n \leq \lim_{n\to\infty} t_n$. ∎

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Corollary (Squeeze Theorem)

Suppose that (s_n) *and* (t_n) *are convergent sequences such that* $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n$ *and, for some other sequence* (x_n), $s_n \leq x_n \leq t_n$.

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$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n.$$

Proof.

Exercise. Note that we must prove (x_n) is convergent. ■

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Proof.

Exercise. ∎

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Proof.

If $t = 0$, $|s_n t_n - 0| \leq |s_n||t_n - 0| + |0||s_n - s| \Rightarrow |s_n t_n| \leq |s_n||t_n|$. Since (s_n) is convergent, there exists some M such that $|s_n| \leq M$ for all $n \in \mathbb{N}$. Since $t_n \to t = 0$, there exists an N such that $n \geq N$ implies $|t_n| < \frac{\varepsilon}{M}$. Then for $n \geq N$, $|s_n t_n| \leq |s_n||t_n| < M\left(\frac{\varepsilon}{M}\right) = \varepsilon$.

If $t \neq 0$, choose N_1 such that $n \geq N_1$ implies $|s_n - s| < \frac{\varepsilon}{2|t|}$. If $t = 0$, this number is not defined, so we should deal with it as a special case.

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If $t \neq 0$, choose N_1 such that $n \ge N_1$ implies $|s_n - s| < \frac{\varepsilon}{2|t|}$. Also choose N_2 such that $n \ge N_2$ implies $|t_n - t| < \frac{\varepsilon}{2M}$ (where M is as above). Set $N := \max\{N_1, N_2\}$ and note that, for $n \ge N$, $|s_n t_n - st| \le |s_n||t_n - t| + |t||s_n - s| < M(\frac{\varepsilon}{2M}) + |t|\frac{\varepsilon}{2|t|} = \varepsilon$. ∎

Theorem (Quotients of Limits)

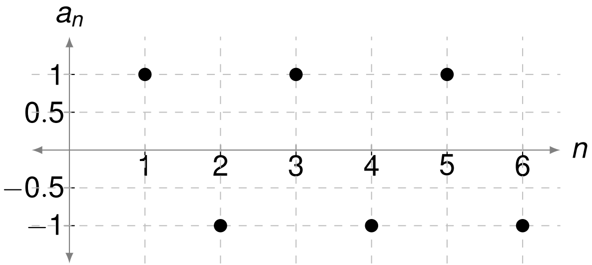
Suppose $s_n \to s$, $t_n \to t$*. Suppose further that* $t_n \neq 0$ *for all* n *and that* $\lim_{n\to\infty} t_n \neq 0$*. Then* $\lim_{n\to\infty}\left(\frac{s_n}{t_n}\right) = \frac{s}{t}$*.*

Proof.

Theorem 2.17 proof in TBB, pages 40-41. ∎

Defining Subsequences

Let's return to a sequence we have seen before:



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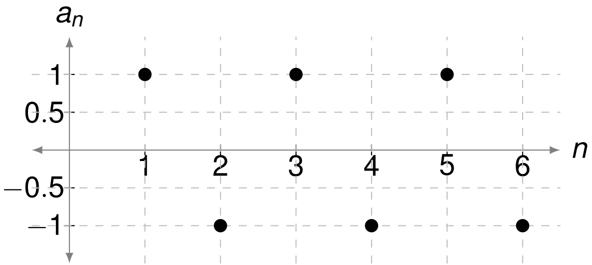

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Proof Idea.

Let $\varepsilon = 1$. We must show for all N and L, there exists $n \geq N$ s.t. $|a_n - L| \geq 1 = \varepsilon$. ■

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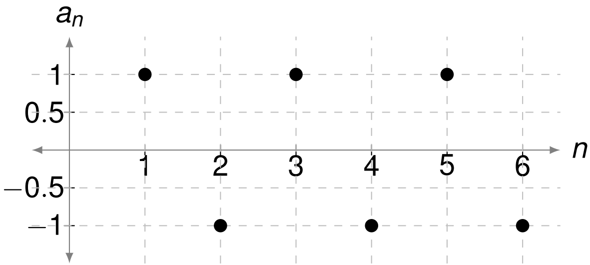

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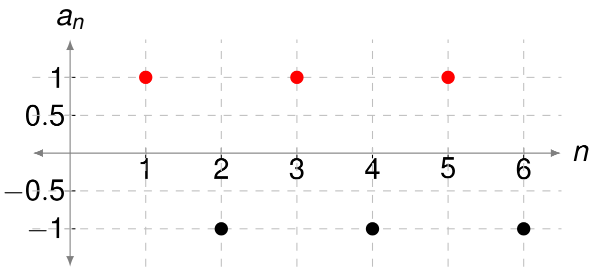
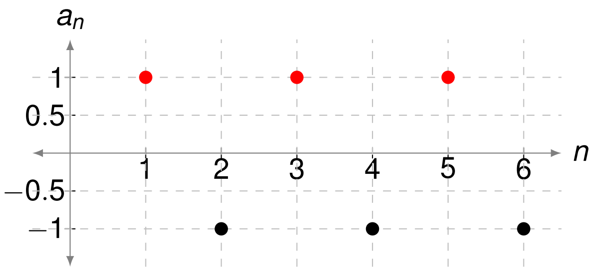


We can try to isolate a **subsequence** (a_{n_k}) - an infinite subset of (a_n) - that *does* converge. For example, if n is odd, $a_n = 1$, so let $n_k = -1 + 2k$. Then $(a_{n_k}) = (1, 1, 1, \dots)$.



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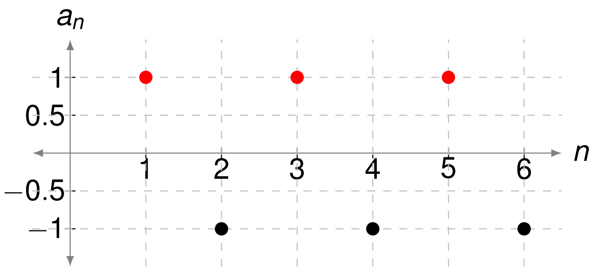

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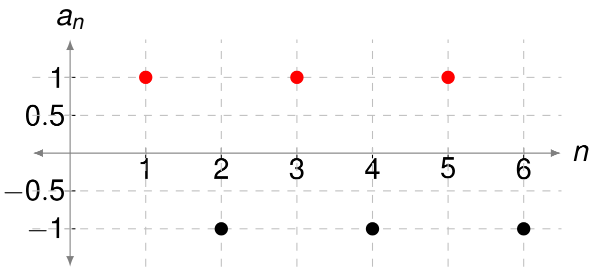
These two subsequences converge to 1 and -1 respectively. These are all of our **subsequential limits**.



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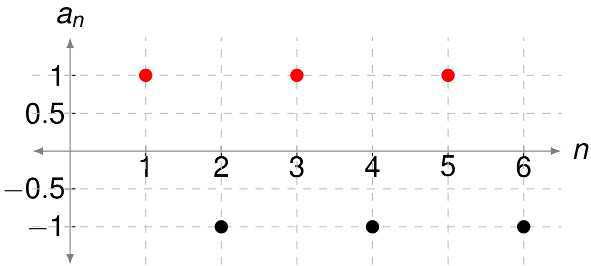

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These two subsequences converge to 1 and -1 respectively. These are all of our **subsequential limits.**The highest of the subsequential limits for a sequence (s_n) is the **limit superior** of (s_n), $\limsup_{n\to\infty} s_n$.The lowest of the subsequential limits for a sequence (s_n) is the **limit inferior** of (s_n), $\liminf_{n\to\infty} s_n$.

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Exercise: Find a monotonic subsequence of the sequence $a_n = (-1)^n n$.

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Monotonic Subsequences

Exercise: Find a monotonic subsequence of the sequence $a_n = (-1)^n n$.
The $(-1)^n$ piece flips the polarity of the sequence each term. Let's just focus on the even terms.
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Let's define $x_m \in (x_n)$ to be a turnback point if no element past x_m is greater than x_m. What happens if there are infinitely many turnback points? We can form an infinite subset of (x_n) consisting of just these points. By their definition, $x_{m_1} \geq x_{m_2}$ if $m_1 < m_2$.

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What if there are finitely many turnback points? Then there is a *farthest* turnback point, x_M. So x_{M+1} is not a turnback point, meaning there is some $m_1 > M+1$ where $x_{m_1} > x_{M+1}$.

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Corollary (Bolzano-Weierstrass Theorem)

Every bounded sequence (x_n) contains a convergent subsequence.

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As niche as this theorem seems, this is a remarkable tool on $\mathbb{R}$ and will come in handy for us later.

Theorem: (x_n) is convergent iff $\limsup_n x_n = \liminf_n x_n$ and these are finite.

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Proof: $(\Rightarrow)$ is left as an exercise.

($\Leftarrow$) Fix $\varepsilon > 0$**. First, if** $\limsup_{n\to\infty} x_n = L$**, then there exists an** N_1 **such that, for all** $n \geq N_1$**,** $x_n - L < \varepsilon$**.**

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Similarly, if $\liminf_{n\to\infty} x_n = L$ **as well, then there exists an** N_2 **such that, for all** $n \geq N_2$**,** $L - x_n < \varepsilon$**.**

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Hence for all $n \geq \max\{N_1, N_2\} =: N$**,**
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By definition, $x_n \to L$**.**

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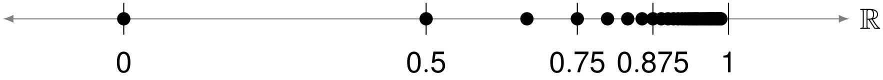
This gives us another way to determine whether $(-1)^{n+1}$ converges: Since $1 = \limsup_n (-1)^{n+1} \neq \liminf_n (-1)^{n+1} = -1$, the sequence does not converge.

“Close" Sequences

We have conceptualized *convergent* sequences as sequences that approach a point. What happens if we think of a sequence where the points get closer to *one another*?



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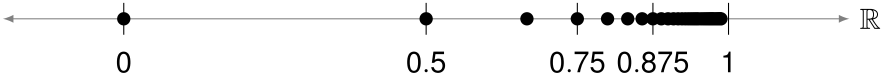

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We say a sequence (s_n) is **Cauchy** if, for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that, whenever $n, m > N$,

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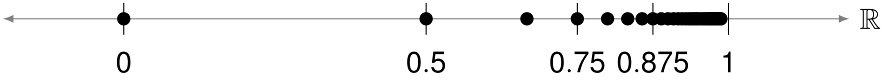

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The biggest thing missing from this definition is its biggest utility - there is no limiting value given for this sequence.

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Question: What is a Cauchy sequence that is not convergent? In $\mathbb{Q}$, there are quite a few. The sequence of decimal approximations for $\sqrt{2}$, $1, 1.4, 1.41, 1.414, \ldots$, has no limit in $\mathbb{Q}$ because it has its limit in a bigger, *complete* space $\mathbb{R}$.

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Theorem (Cauchy Criterion)

A sequence (s_n) is convergent iff (s_n) is Cauchy.

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Notions of equivalence are very strong in analysis - we are saying that these notions of “closeness" (Cauchy) and “limiting" (convergent) are identical in $\mathbb{R}$ (whereas they are not the same in $\mathbb{Q}$!).

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Types of Sums

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A **finite sum** is a sum over a finite index $k \in \{1, \ldots, n\}$. For example,

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

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We can take the terms of this sum and cancel them out, link-by-link: $(\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots$. This is known as a **telescoping sum**.

Types of Sums

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A **finite sum** is a sum over a finite index $k \in \{1, \ldots, n\}$. For example,

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

A special type of finite sum is

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1} = 1 - \frac{1}{n+1}.$$

We can take the terms of this sum and cancel them out, link-by-link: $(\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots$. This is known as a **telescoping sum**.

There are also **geometric sums**:

$$1 + r + r^2 + r^3 + \cdots + r^{n-1} + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

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We can attempt to take limits in each of these sums in the variable n to make infinite ordered sums, or **series.** Each **partial sum** becomes an element of a sequence: $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_k$, where

$$(s_k) = (a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots).$$

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We get plenty of help on series from our study on sequences:

If $\sum_{k=1}^{\infty} a_k$ converges, the sum is unique.

If $\sum_{k=1}^{\infty} a_k = a$ and $\sum_{k=1}^{\infty} b_k = b$, then $\sum_{k=1}^{\infty}(a_k + b_k)$ converges to $a + b$.

If $\sum_{k=1}^{\infty} a_k = a$, then $\sum_{k=1}^{\infty} ca_k$ converges to ca.

If $\sum_{k=1}^{\infty} a_k = a$ and $\sum_{k=1}^{\infty} b_k = b$ and $a_k \le b_k$ for all k, then $a \le b$.

We also get a theorem we haven't had much use for but comes in handy when discussing series:

Theorem

Let $M \geq 1$ be any integer. Then the series $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$ converges iff the series $\sum_{k=1}^{\infty} a_{M+k} = a_{M+1} + a_{M+2} + a_{M+3} + \cdots$ converges.

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We call the notion of a sequence beginning beyond the first element the "tail end" of a series. The above theorem says what happens at the beginning of a series doesn't impact much when it comes to convergence - only what happens as the series goes toward infinity.

Calculating with Sums

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The first sum gives us $1 + \frac{1}{2}$, and the second sum gives $-\frac{1}{3} - \frac{1}{4}$. (Why?) Adding these and multiplying by $\frac{1}{8}$ gives us $\frac{11}{96}$.

Series Tests

Theorem (Ratio Test)

Let $\sum x_n$ *be a series,* $x_n \neq 0$ *for all n, such that*

$$L := \limsup_n \frac{|x_{n+1}|}{|x_n|} \quad \textit{exists.}$$

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1. *If $L < 1$, then $\sum x_n$ converges absolutely.*
2. *If $\frac{|x_{n+1}|}{|x_n|} > 1$ for all $n > N$ for some $N \in \mathbb{N}$, then $\sum x_n$ diverges.*

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If $\sum x_n$ *converges, then* $x_n \to 0$.

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The rest uses the techniques we have developed and is left as a homework exercise.

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Example: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, but $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converges.

Proof of Alternating Series Test.

If s_n is the nth partial sum of this series, we note that s_{2n} is decreasing since (x_n) is decreasing (and hence $x_{i-1} \leq x_i$):

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Prelude to Topology

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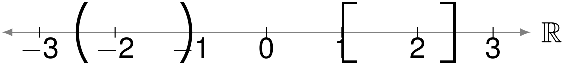
We can now attempt to understand the **topology of sets** in $\mathbb{R}$, which measures closeness and separation between elements and sets in a space. In this course we will define what open and closed sets are in terms of their points - this is known as **point-set topology**.



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Question: What points in an open interval are interior points? **All of them!** If $x \in (a, b)$, pick the smaller of $x - a$ and $b - x$ - call it ε. Note these can’t be zero since $a, b \notin (a, b)$. Finally, $(x - \varepsilon, x + \varepsilon) \subset (a, b)$. So any arbitrary $x \in (a, b)$ is an interior point.

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2. What points in $[a, b]$ are interior points? **All points other than a and b.**

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4. What points in $\mathbb{Q}$ are interior points? **None.** Every interval of $\mathbb{R}$ contains both rational and irrational points. (We will soon see that points in $\mathbb{Q}$ *aren't* isolated...)

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Let E be a set of real numbers. Any point x that belongs to E is said to be an **isolated point** of E provided that for some interval $(x - \varepsilon, x + \varepsilon)$,

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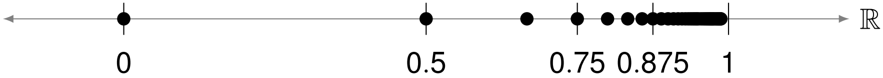

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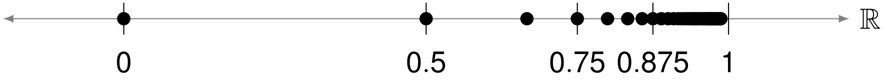

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The set in the definition above should be understood in the context of the picture. The number 1 is not in the set $\{1 - \frac{1}{n} : n \in \mathbb{N}\}$. However, no matter how far you widen your neighborhood starting at 1, you will always have a point in the set. So 1 is an accumulation point.

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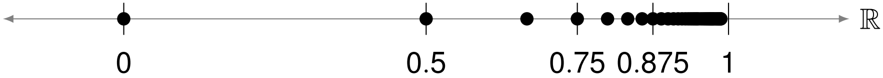

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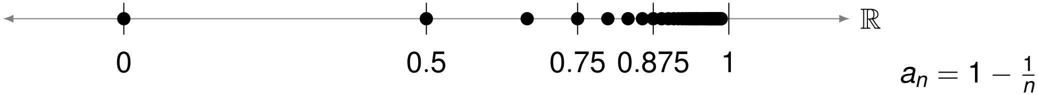

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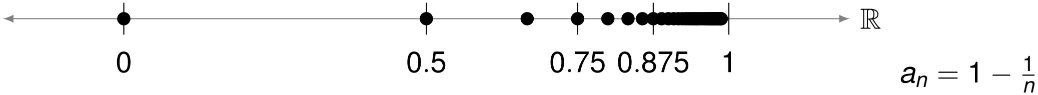

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Closed Sets

Recall the example where adding the accumulation points of (a, b) to the set itself gives us its “closure": $[a, b]$. We would like to formalize this notion. First, let’s define the word “closed" to apply to more than just intervals:

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Claim

All nonempty open sets are of the form (a, b) for some $a < b \in \mathbb{R}$.

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Let G be a nonempty open set of real numbers. Then there exists $a_1 < b_1 = a_2 < b_2 = a_3 < b_3 = \cdots$ such that

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Properties of Open and Closed Sets

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We start off with a banger:

Theorem

Let A be a set of real numbers. Recall that the ***complement*** *of A is the collection of points not in A, written as A^c or $\mathbb{R} \setminus A$. Let $B := A^c$. Then A is open iff B is closed.*

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We will prove both ways by contradiction.
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Corollary

Let $B_1, B_2, B_3, \ldots, B_n$ be closed sets. Then $\bigcup_{i=1}^{n} B_i$ is also closed.

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Compactness

There is one more property of a set that is discovered around a large variety of different branches of analysis. The term “compact" is a rank of strength just below “finite". If something holds for every element of a finite set, there is almost certainly something very strong that can be said about the entire set itself.

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This isn't very strong - it makes sense that boundedness on each point of a finite set implies boundedness. We also certainly require finiteness in order to take a the maximum of the M_i above, so it is unclear how to make this stronger.

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If E is finite and f is bounded on each $x \in E$, then f is (globally) bounded on E.

What if we strengthened our requirements for f?

Definition

We say a function f is **locally bounded** on a set E if, for all $x \in E$, there exists a $\delta > 0$ such that f is bounded on the set $(x - \delta, x + \delta)$.

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This says we can put a smidge of cushion around each point in E and still find a bound for our function on each set. This is a stronger requirement on f; boundedness on a single point is weaker than boundedness on an entire interval.

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It turns out that, with this added strength, we can relax our condition E to be closed and bounded (which we will call **compact**):

Theorem

Let E be closed and bounded. Then every function $f : E \to \mathbb{R}$ that is locally bounded on E is (globally) bounded on E.

Revisiting Bolzano-Weierstrass

Because compactness is so ubiquitous, we look for all things that can imply compactness - or for equivalent definitions to “closed and bounded". One candidate is the **Bolzano-Weierstrass property**.

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Definition

We say a set E has the **Bolzano-Weierstrass Property** if every sequence (s_n) in E has a subsequence converging to a point in E.

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($\Rightarrow$) Let E be closed and bounded and let (x_n) be a sequence contained in E. Since E is bounded, (x_n) is bounded too. We know from another Bolzano-Weierstrass Theorem that any bounded sequence has a convergent subsequence - let (x_{n_k}) be that convergent subsequence and assume it converges to x.

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Is x in E? The definition of limit says that x_{n_k} is eventually $(x - \varepsilon, x + \varepsilon)$. Since x_{n_k} is in E, we see that x is an accumulation point of E by definition. So $x \in E$.

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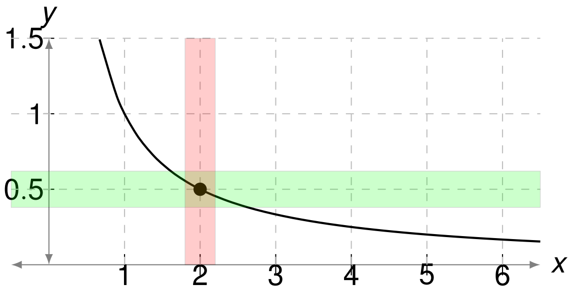

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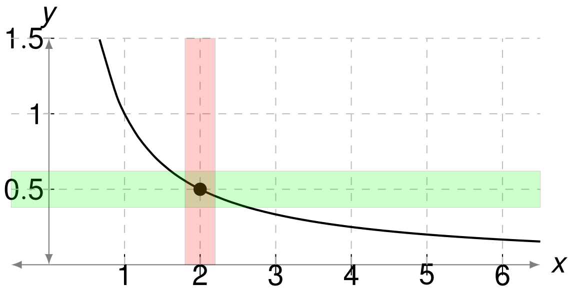

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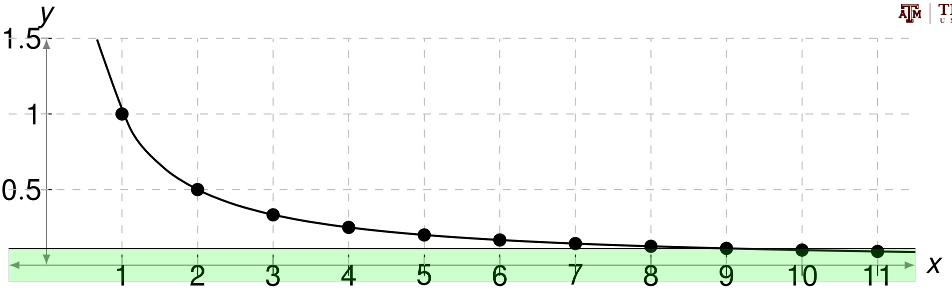

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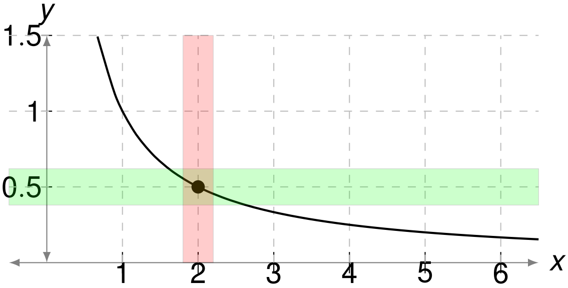
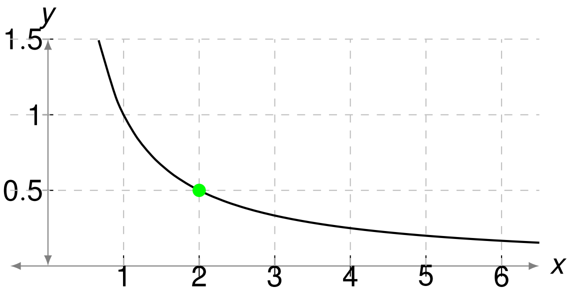


$\lim_{x\to 2} \frac{1}{x} = \frac{1}{2}$

Let's return to the function picture we saw at the beginning of Chapter 2. What we were introducing was an ε-δ definition of a limit. Like sequences, we need something to be satisfied for *any* ε-window around our limiting value. For sequences, we need to go far enough along the x-axis to find an N such that any $n \geq N$ would be in this ε-window. This makes sense since we are finding limits as $n \to \infty$. However, here we are finding a limit as x approaches a finite value. This means that, as x approaches our finite value of 2 in a δ-window, our y-values must be similarly approaching $\frac{1}{2}$.

https://www.desmos.com/calculator/iejhw8zhqd



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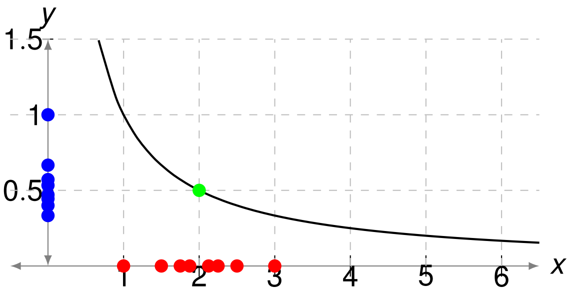

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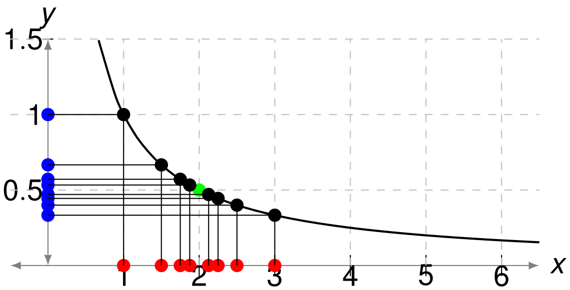

This means that, as x approaches our finite value of 2 in a δ-window, our y-values must be similarly approaching $\frac{1}{2}$. Can you think of a way we could do this with sequences? A picture is worth a thousand words :) but we will use words anyway because it will help us prove strong theorems.

The ε-δ Definition of a Limit

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Note that the definition prohibits x from equalling x_0 in the calculation.This is because a limit doesn't look at the function value at the point itself - only the values surrounding the point.

$$f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

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As with our sequential limit problems, we begin by working backwards. Combining like terms, we see our goal is $|10x - 50| < \varepsilon$. We want to aim with something dealing with $|x - 5|$.

Theorem: $\lim_{x\to 5}(10x - 11) = 39$.
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Choose $\delta = \frac{\varepsilon}{10}$.

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Good news! We can just factor out a 10 and divide. Then $|x - 5| < \frac{\varepsilon}{10}$.

Theorem: $\lim_{x\to5}(10x-11)=39$.

Proof: Let $\varepsilon>0$.

Choose $\delta=\frac{\varepsilon}{10}$.

Then whenever $0<|x-5|<\delta$**, we have** $|x-5|<\frac{\varepsilon}{10}\Rightarrow 10|x-5|<\varepsilon\Rightarrow |(10x-11)-39|<\varepsilon$**, as desired.**

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For example, say we force $|x - 3| < 1$. How big can $|x + 3|$ be?

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Notably, all of the functions we have chosen to evaluate so far are *continuous* functions. We know but have not proven that direct substitution gives their limits. So we wish to define continuous functions so we can use this functionality.

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Notably, all of the functions we have chosen to evaluate so far are *continuous* functions. We know but have not proven that direct substitution gives their limits. So we wish to define continuous functions so we can use this functionality. To do this, it is useful to give one more definition of a limit.

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We have now defined the same notion of a functional limit in two different ways. In order for what we have just done to make sense, these two definitions must be equivalent. We prove that now.

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Limit Laws, Revisited

Thanks to this second definition, a lot of our laws for sequences carry directly over to functions. For example:

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- **(Squeeze Theorem)** Suppose $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = L$. If $h : E \to \mathbb{R}$ is a function such that

  $$f(x) \leq h(x) \leq g(x)$$

  for all $x \in E$ except perhaps at $x = x_0$, then $\lim_{x\to x_0} h(x) = L$.

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The issue is that f is *discontinuous at* x. That is, even if we take a sequence $x_n \to x$, $f(x_n)$ does not necessarily converge to $f(x)$.

Definition (Sequential Definition of Continuity)

Let $f : (a, b) \to \mathbb{R}$ be a function, and let $x_0 \in (a, b)$. We say that f is **continuous at x_0** if $\lim_{x\to x_0} f(x) = f\left(\lim_{x\to x_0} x\right) = f(x_0)$.

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Theorem

Let F be a function that is continuous at the point L. Then if $\lim_{x\to x_0} f(x) = L$, we have

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Let $f : [a, b] \to \mathbb{R}$ be a function and let $x_0 \in [a, b]$. Then we say that f is **continuous** at x_0 if, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \varepsilon$.

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- The composition of these functions on the intersection of their domains (we will prove this later, but you may have a hint as to why).

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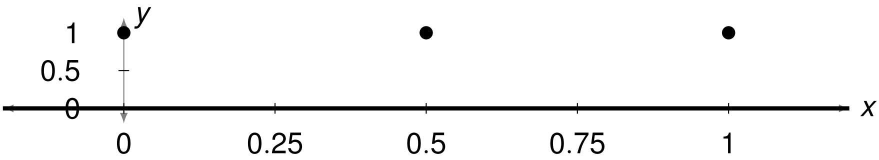

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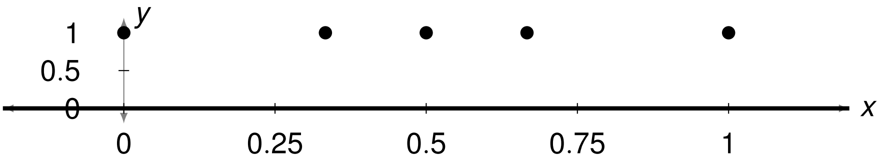

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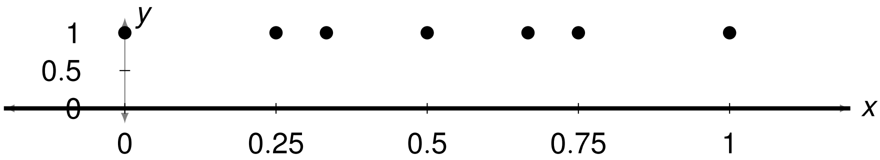

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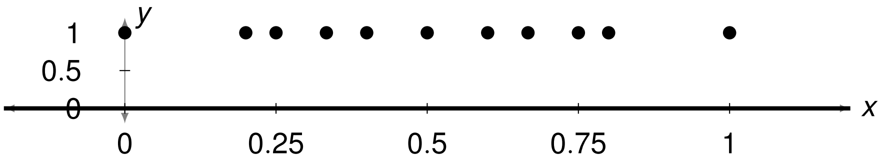

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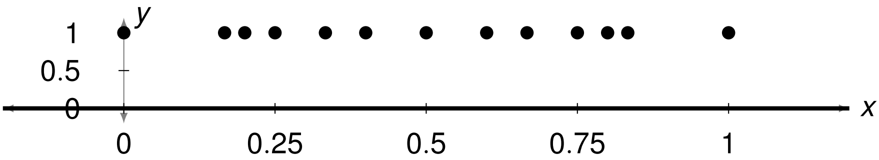

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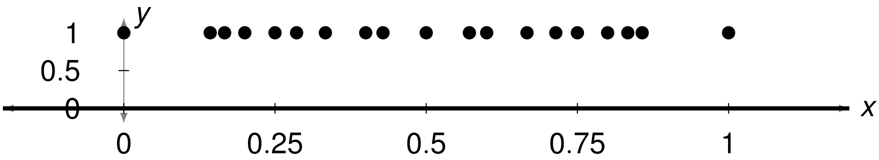

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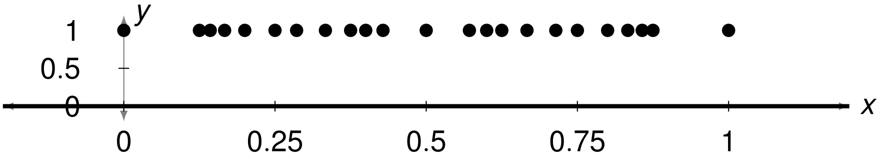

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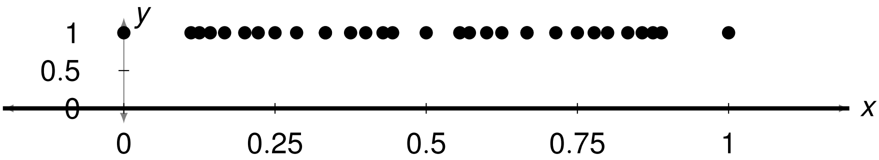

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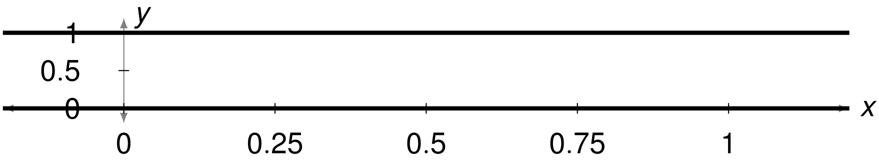

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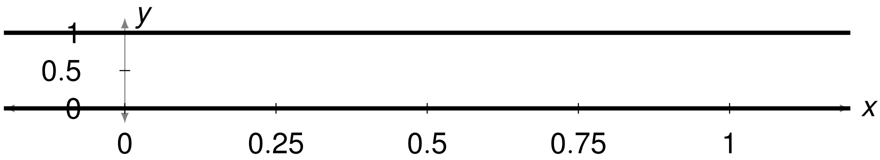

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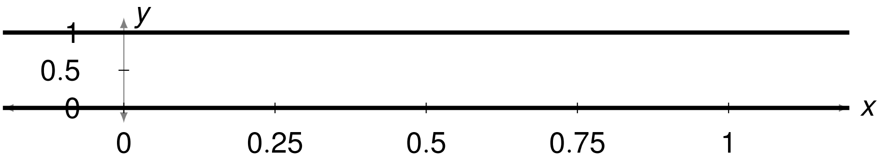

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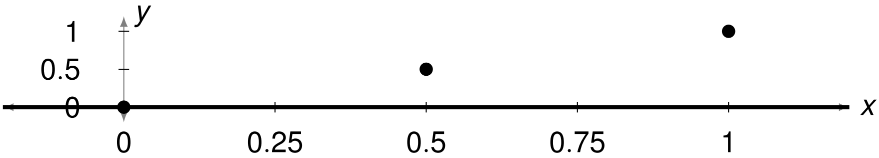

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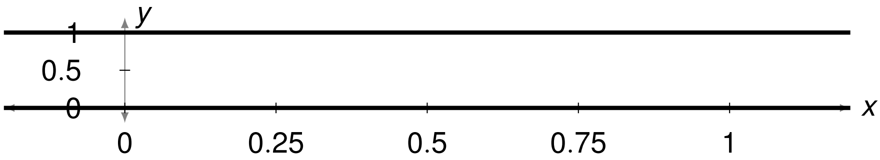

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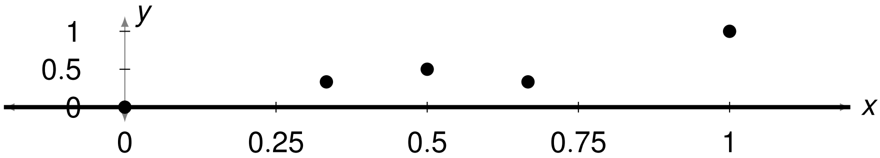

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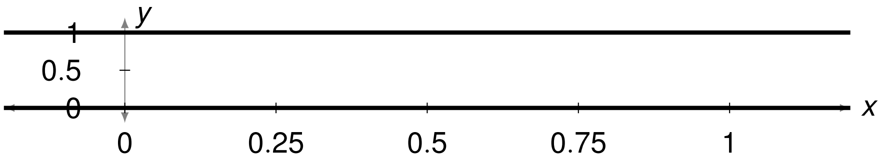

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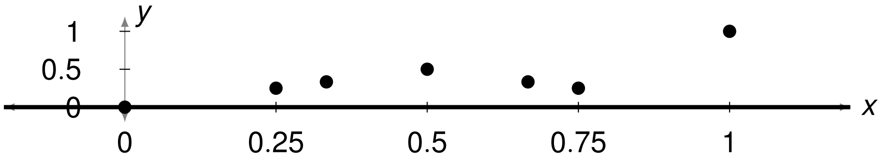

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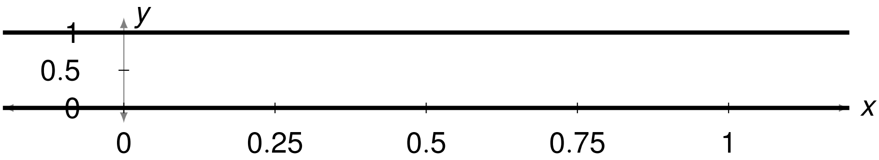

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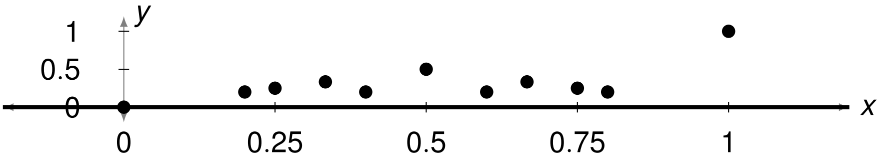

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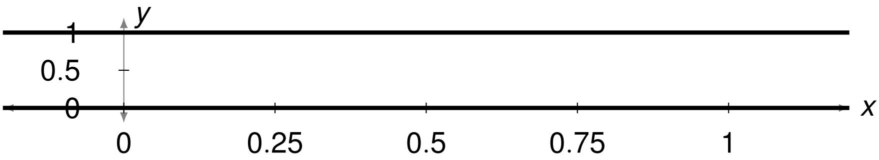

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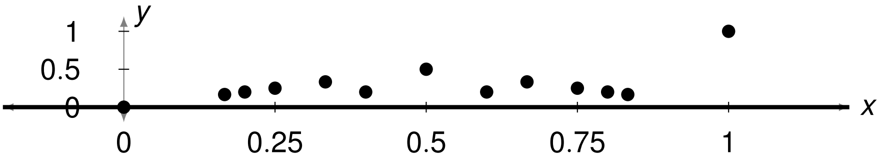

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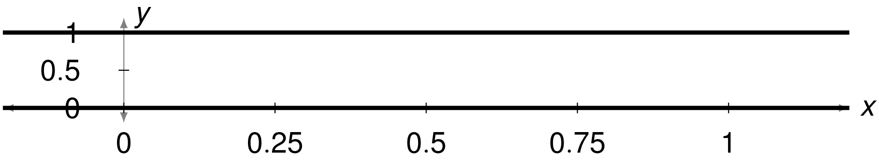

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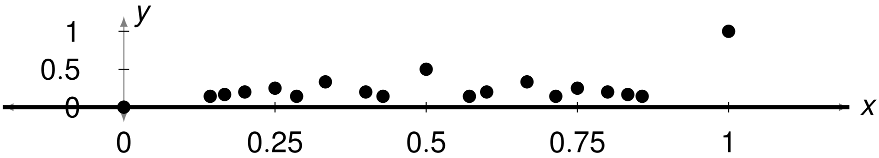

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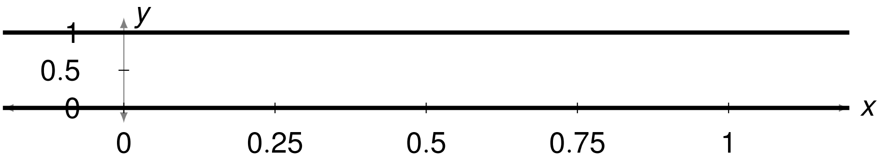

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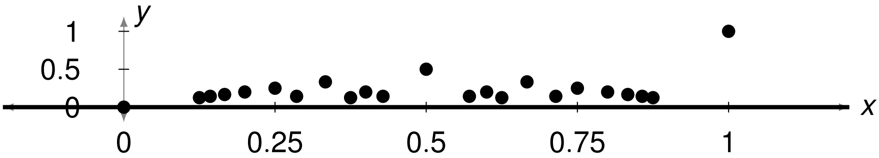

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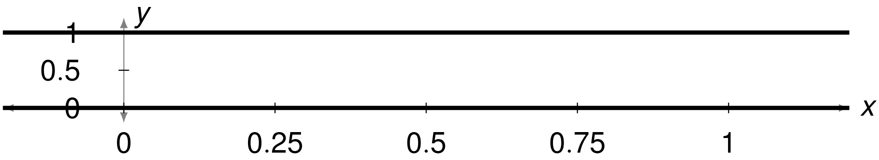

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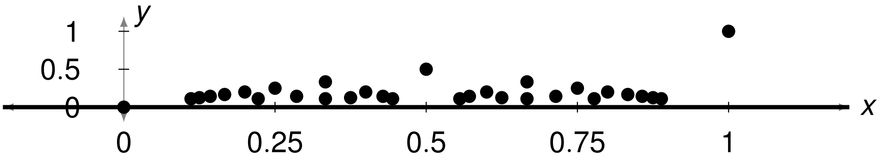

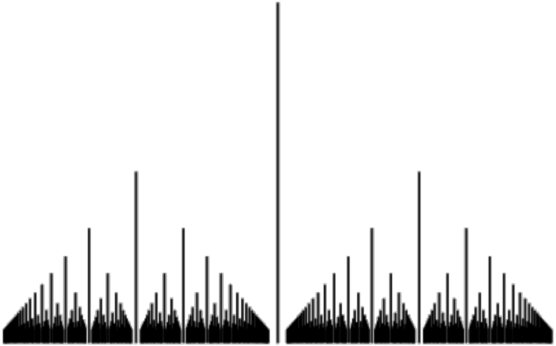

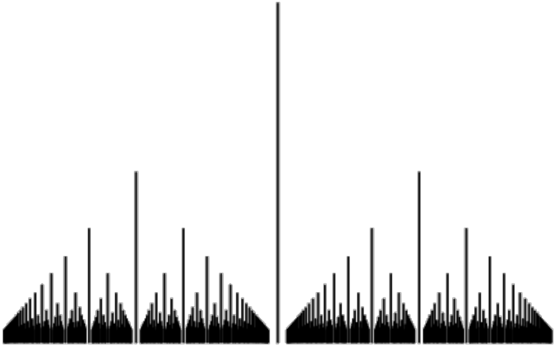

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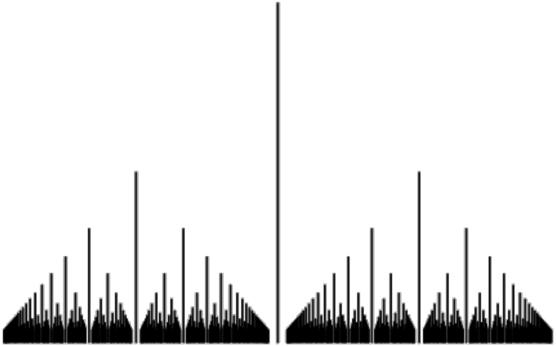

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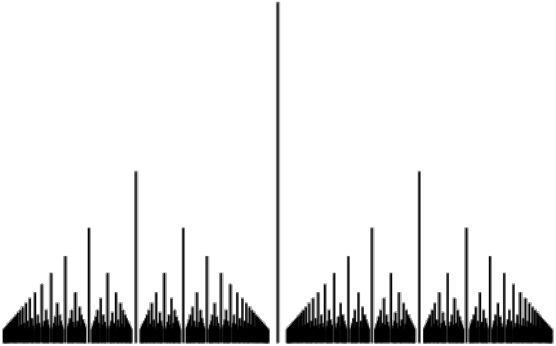

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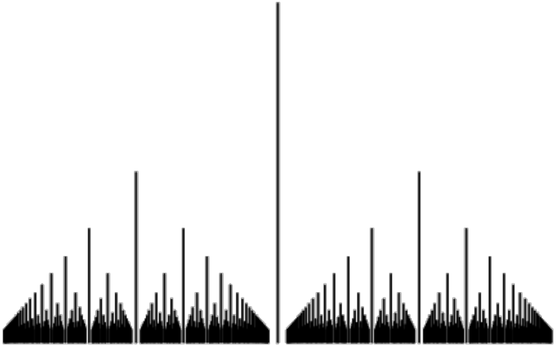

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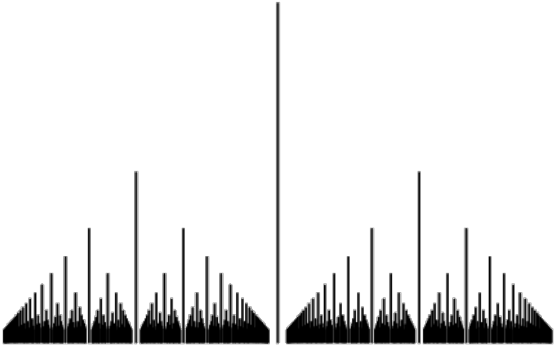

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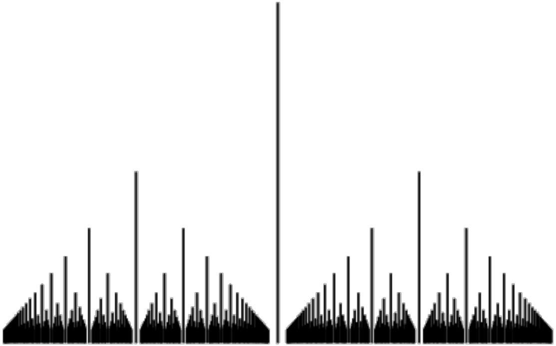

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Let $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$ and suppose that $f(A) \subset B$. Suppose that f is continuous at a point $x_0 \in A$ and that g is continuous at the point $y_0 = f(x_0) \in B$. Then the composition function $g \circ f : A \to \mathbb{R}$ is continuous at x_0.

Uniform Continuity

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Let's write the definition of continuity one more time. We will make a slight change to the domain for a future application.

Definition (ε-δ Definition of Continuity on an Interval I)

Let $f : I \to \mathbb{R}$ for an interval $I \subset \mathbb{R}$. Then we say that f is **continuous** on I if, for all $x_0 \in I$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \varepsilon$.

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In quantifiers,

$$\forall x_0 \in I \, \forall \varepsilon > 0 \, \exists \delta > 0 \ni, \forall x \ni |x - x_0| < \delta, |f(x) - f(x_0)| < \varepsilon.$$

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This means that, before proceeding with finding a δ for our proof, we get to work with a prescribed $x_0 \in I$ as well as some $\varepsilon > 0$ (as long as these are arbitrary prescriptions).

Theorem: $f(x) = 10x - 11$ is continuous at $x = 5$.

Proof: Let $\varepsilon > 0$.
Choose $\delta = \frac{\varepsilon}{10}$.
Then whenever $|x - 5| < \delta$, we have $|x - 5| < \frac{\varepsilon}{10} \Rightarrow 10|x - 5| < \varepsilon \Rightarrow |(10x - 11) - 39| < \varepsilon$, as desired. (Note $f(5) = 39$.)

Let's observe the proof that a familiar function is continuous at a familiar point. The proof itself looks nearly identical - we only removed the condition that $0 < |x - 5|$.

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Theorem: $f(x) = x^2$ is continuous at $x = 3$.

Proof: Let $\varepsilon > 0$.
Choose $\delta = \min\{1, \frac{\varepsilon}{7}\}$.
Then whenever $0 < |x - 3| < \delta$, we have
$|x - 3| < 1$, so $|x + 3| \leq |x - 3| + |6| < 7$.
Hence $|x^2 - 9| = |x - 3||x + 3| < 7|x - 3|$.
Since $|x - 3| < \frac{\varepsilon}{7}$, $|x^2 - 9| < \varepsilon$ as desired.
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Let's observe the proof that a familiar function is continuous at a familiar point. The proof itself looks nearly identical - we only removed the condition that $0 < |x - 5|$. Here's another proof with a similar adaptation. Notice that we had to engage the problem slightly differently than the previous one. Why?

Would we have to do something different with either problem should $x = 4$? The $10x - 11$ problem can be left untouched - our choice of δ *does not depend on our choice of* x. But our x^2 problem must be changed, since our choice of δ *depends on* x.

Continuity of f at $x_0 \in I$:

$$\forall x_0 \in I \, \forall \varepsilon > 0 \, \exists \delta > 0 \ni, \forall x \ni |x - x_0| < \delta, |f(x) - f(x_0)| < \varepsilon.$$

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Definition

Let f be defined on a set $A \subset \mathbb{R}$. We say that f is **uniformly continuous** on A if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $x, y \in A$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

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Notice that this is the property of a function and a *set*, rather than of a function and a point.

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Theorem

If a function f is uniformly continuous on a bounded interval I, then f is bounded on I.

Proof.

Let $a := \min I$, $b := \max I$. By uniform continuity, we can choose a $\delta > 0$ such that $|f(y) - f(x)| < 1$ whenever $x, y \in I$ and $|x - y| < \delta$.

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Proof.

Combine the previous two theorems. Since f is uniformly continuous on $[a, b]$, which is a bounded interval, it is bounded on $[a, b]$. ■

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We will use the fact that $[a, b]$ is closed and bounded, and hence compact. Assume BWOC that f is not uniformly continuous. Then the negation of this statement implies that there are sequences (x_n) and (y_n) such that $x_n - y_n \to 0$ but $|f(x_n) - f(y_n)| \geq \varepsilon$ for some $\varepsilon > 0$. (This will be an exercise.)

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Corollary (Existence of Absolute Max/Min)

Let f be continuous on $[a, b]$. Then f possesses both an absolute maximum and an absolute minimum.

Proof.

Let $M = \sup\{f(x) : x \in [a, b]\}$.

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Whenever S is unbounded below, we say $-\infty$ is a **cluster point** of S.

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This is reminiscent of our definition of a sequential limit.

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We will return to this discussion when we discuss continuity and topology.

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We can do a similar thing for $L < 0, M \in \mathbb{R}$ by finding an n such that $2n + 1/2 > M$. Since this is true for any $n \in \mathbb{N}$, there is no $M \in \mathbb{R}$ such that $x \geq M$ implies $|f(x) - L| < 1$ for any L.

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This makes a difference because the sequence $\sin(\pi n)$ converges to 0 as $n \to \infty$.

Lemma

Suppose $f : S \to \mathbb{R}$ *is a function,* ∞ *is a cluster point of* $S \subset \mathbb{R}$*, and* $L \in \mathbb{R}$*. Then*

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Proof.

The proof is similarly conducted to how one proves the statement adapted as $x \to c$. ∎

Monotonic Functions

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Definition

We say a function f is **increasing** [resp. **decreasing**] on an interval I if $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$ [resp. $f(x_1) \geq f(x_2)$]. We say the function is **strictly increasing** [resp. **strictly decreasing**] on I if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ [resp. $f(x_1) > f(x_2)$].

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Consider the functions $g = \frac{f+|f|}{2}$ and $h = \frac{f-|f|}{2}$.

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Let $(x_j)_{j \in J}$ be all the points at which f is discontinuous. Each interval I_j caused by this continuity at a point $x_j \in I$ as positive length. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, each I_j has a distinct rational point contained in it. That rational point is not in any other interval since the intervals are disjoint.

Let $\varphi : J \to \mathbb{Q}$ be the function mapping each element of the index set $j \in J$ to the rational point in the interval I_j. Then by our discussion φ is injective. So $|J| \leq |\mathbb{Q}|$ and J is countable. ∎

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Lemma

A monotonic function on an interval $f : [a, b] \to \mathbb{R}$ is bounded.

Proof.

Exercise. ∎

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WLOG let f be increasing. From our reading we recall that monotonic functions only have jump discontinuities. If x_0 is a point of discontinuity on I, then the interval

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Note the collection $(S_n)_{n \in \mathbb{N}}$ is countable. A countable union of countable sets is countable, and these sets are in fact finite. So $\bigcup_{n \in \mathbb{N}} S_n = I$ is countable. ∎

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Intuitively, the reason there can be no jump discontinuities for cluster points is that the domain of f (now the range of f^{-1}) has no gaps. This is formalized in Lebl. ∎

Revisiting the Derivative

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Definition

Let f be defined on an interval I and let $x_0 \in I$. The **derivative** of f at x_0, denoted by $f'(x_0)$, is defined as

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If $f'(x_0)$ is finite we say that f is **differentiable** at x_0. If f is differentiable at every point of a set $E \subset I$, we say f is differentiable on E. When E is all of I, we say f is a **differentiable** function.

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Theorem (Sum/Constant Multiple Rules)

Let f, g be defined on an interval I and let $x_0 \in I$. If f, g are differentiable at x_0 then so are cf and $f + g$. Furthermore, $(cf)'(x_0) = cf'(x_0)$, and $(f + g)'(x_0) = f'(x_0) + g'(x_0)$.

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Exercise. ∎

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So we get $h'(x_0) = \lim_{x\to x_0} f(x)g'(x_0) + g(x_0)f'(x_0)$, hence the theorem. ∎

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With I, f, g, x_0 *as in previous theorems, if* $g(x_0) \neq 0$*, then* $\frac{f}{g}$ *is differentiable at* x_0*. Furthermore,* $(\frac{f}{g})'(x_0) = \frac{g(x_0)f'(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}$*.*

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Letting $x \to x_0$, we get the desired result. ∎

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Let I_1, I_2 be intervals. Suppose $f : I_1 \to I_2$ is differentiable at $x_0 \in I_1$ and $g : I_2 \to \mathbb{R}$ is differentiable at $f(x_0)$. Then the composite function $h = g \circ f$ is differentiable at x_0, and $h'(x_0) = g'(f(x_0))f'(x_0)$.

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Observing that the second factor converges to 0 as $n \to \infty$ since $f'(x_0) = 0$, we are done. ∎

Theorem (Power Rule, Kind Of)

Let $n \in \mathbb{N}$. Then $f(x) = x^n$ is differentiable, and $f'(x) = nx^{n-1}$.

Proof.

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We use the **Inverse Function Theorem** to claim an inverse for x^n exists, which we cannot prove now - however, the proof will not rely on the Power Rule, so this is okay. Write $g(x) = x^{1/n}$ and $f(x) = x^n$. Then $g(f(x)) = x$, so by the Chain Rule $g'(f(x))f'(x) = 1$.

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Theorem (Power Rule, Even More Kind Of)

Let $f(x) = x^{m/n}$ for integers m/n. Then f is differentiable on the interior of its domain, and $f'(x) = \frac{m}{n}x^{\frac{m}{n}-1}$.

Proof.

Exercise. ∎

Maxima and Minima

Derivatives have a few main applications: the existence of local maxima and minima, the Mean Value Theorem, L'Hôpital's Rule, and Taylor polynomials.

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Hence $f'_+(x_0) \leq 0$ and $f'_-(x_0) \geq 0$.

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Hence $f'_+(x_0) \leq 0$ and $f'_-(x_0) \geq 0$. If $f'(x_0)$ exists, then these one-sided limits are equal to it, so $f'(x_0) = 0$. ■

Mean Value Theorem

Theorem (Rolle's Theorem)

Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then there exists $c \in (a, b)$ such that $f'(c) = 0$.

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Suppose that f is a continuous function on the closed interval $[a, b]$ *and differentiable on* (a, b)*. Then there exists* $c \in (a, b)$ *such that*

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If $f'(c) \geq 0$, This means $f(x_2) \geq f(x_1)$. This is the definition of non-decreasing. ∎

L'Hôpital's Rule

Before giving insight into L'Hôpital's Rule, we need to generalize the Mean Value Theorem:

Theorem (Cauchy Mean Value Theorem)

Let f and g be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that

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Proof.

This is an exercise, but the hint is to consider the function
$\varphi(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$. ∎

Let's discuss why L'Hôpital's Rule *should* work. Recall:

Theorem (L'Hôpital's Rule: $\frac{0}{0}$ Form)

Suppose that f and g are differentiable in an interval (c, d) containing a except possibly at a. If $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$,

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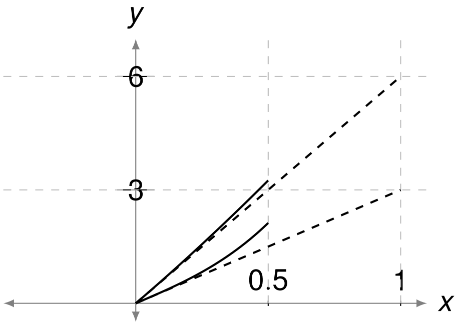
So why is it that this theorem should be true? It may be useful to think of functions in terms of their Taylor series approximations at the point a. Here are the functions $6x + x^2$ and $3x + 5x^3$ - their asymptotic behavior close to $x = 0$ looks like $6x$ and $3x$ respectively.



Algebraically speaking, if $f(a) = 0 = g(a)$,

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Taking a limit as $x \to a$, assuming f'/g' is continuous, we get the theorem statement in L'Hôpital. This is not the way we will prove this theorem, as it assumes too many things about the functions f and g, like that $f(a), g(a)$ exist, that $g(x) \neq 0$ in this neighborhood, and that f'/g' is continuous. However, it is a helpful way to understand the theorem.

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Suppose that f and g are differentiable in a neighborhood N of $x = a$ except possibly at $x = a$. If (i,ii) $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$, (iii) $g'(x) \neq 0$ in N, and (iv) $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

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Taylor Polynomials

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In Calculus we are introduced to Taylor series as *vast* improvements on linear approximations $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$. It's a beautiful result that allows us to find values of a smooth function based on very local data and limits.

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Let f possess at least $n+1$ derivatives on an open interval I and let $c \in I$. Let $R_n(x) = f(x) - P_n(x)$, where $P_n(x)$ is the nth Taylor polynomial.

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Before moving onto the proof, let's see an example to understand this new functionality.

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Not only is $|\sin(x)| \leq 1$; in fact $|f^{(n)}(x)| \leq 1$ for all n! So we can successfully bound $R_n(x)$ with $\frac{|x^{n+1}|}{(n+1)!}$. If we wished to approximate $\sin(x)$ on, say, $[-1, 1]$ up to a certain error, we could even use this formula to find the lowest degree of P_n we could use to achieve this.

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We have mentioned inverse functions briefly in this course. Recall that a function composed with its inverse function in either order comes out to be the function $f(x) = x$.

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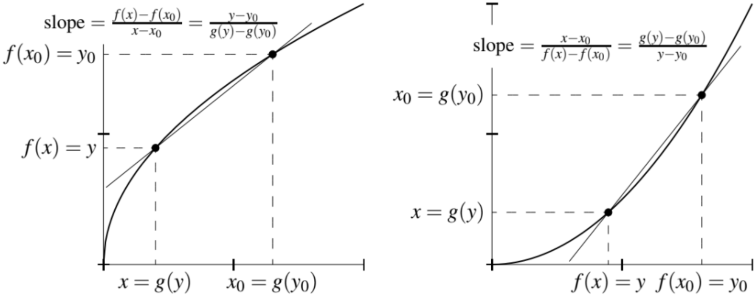

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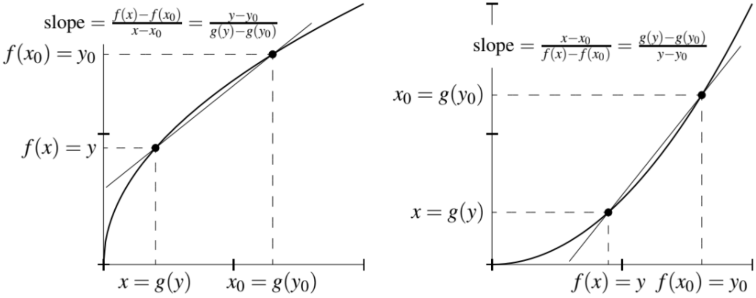

Note that slopes are also inverted on nice domains such as this one. But we run into issues inverting functions like $f(x) = x^3$. (Why?)

Lemma

Let $I, J \subset \mathbb{R}$. If $fI \to J$ is strictly monotone (hence one-to-one), onto, differentiable at $x_0 \in I$, and $f'(x_0) \neq 0$, then the inverse f^{-1} is differentiable at $y_0 = f(x_0)$ and

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If f is continuously differentiable and f' is never zero, then f^{-1} is continuously differentiable.

Proof Idea.

By our discussion regarding continuity of inverse functions back before discussing derivatives, f has a continuous inverse; let's call it $g : J \to I$. Pick an $x \in I$; then define $y := f(x)$.

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The rest comes from checking to ensure our limits as x goes to x_0 (for the function f) translate nicely to limits as y goes to y_0 (for the function f^{-1}) so that once can take the limit. This works nicely since f is continuous (it is differentiable) as is f^{-1} at this point.

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Theorem (Inverse Function Theorem)

Let $f : (a, b) \to \mathbb{R}$ be a continuously differentiable function, $x_0 \in (a, b)$ a point where $f'(x_0) \neq 0$. Then there exists an open interval $I \subset (a, b)$ with $x_0 \in I$, the restriction $f|_I$ is injective with a continuously differentiable inverse $g : J \to I$ defined on the interval $J := f(I)$, and

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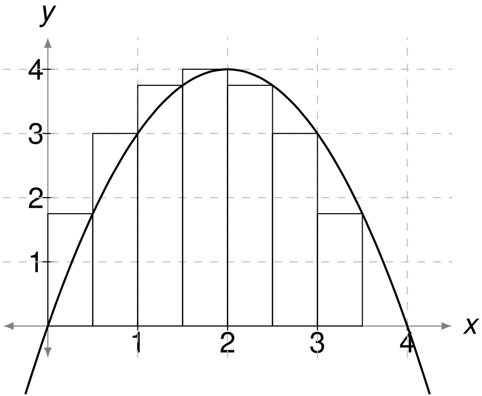

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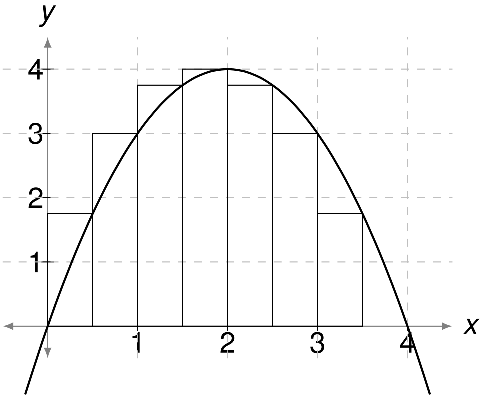

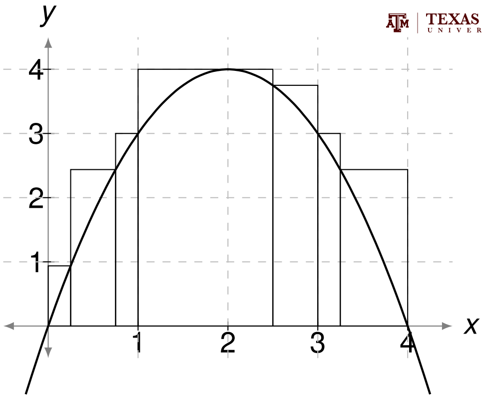

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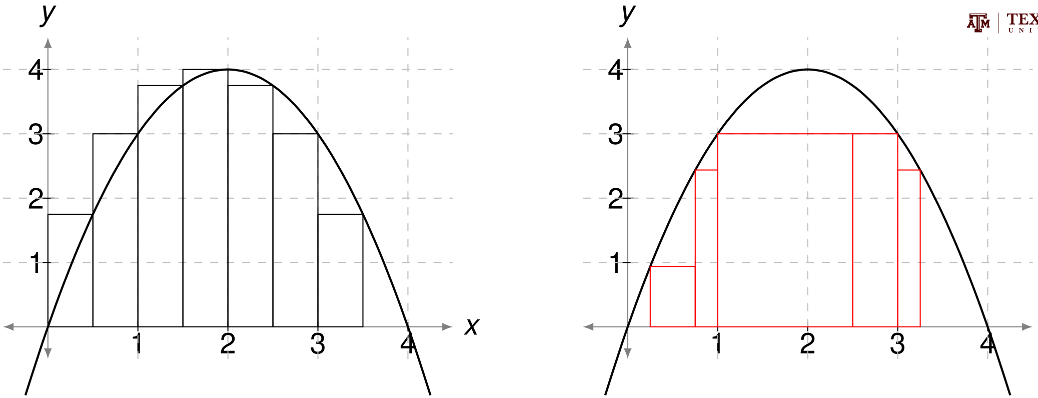

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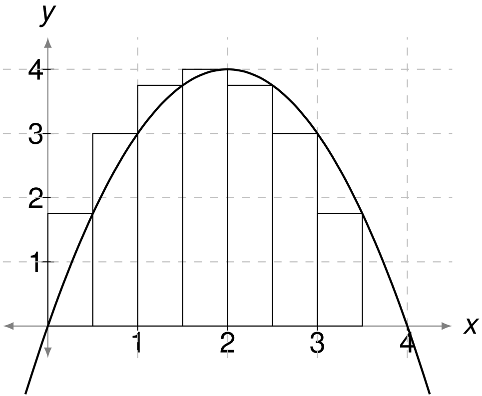

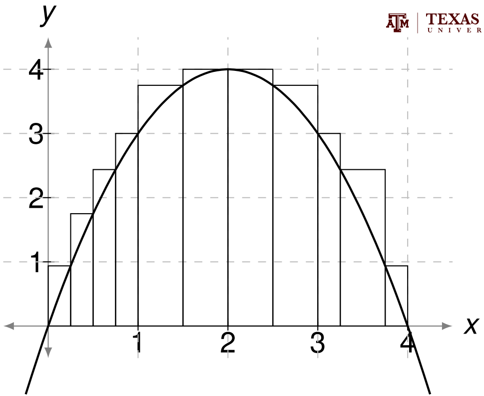

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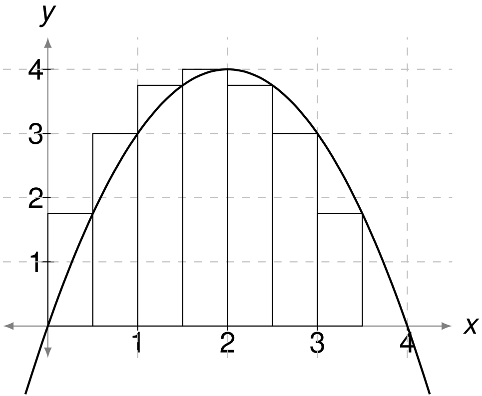

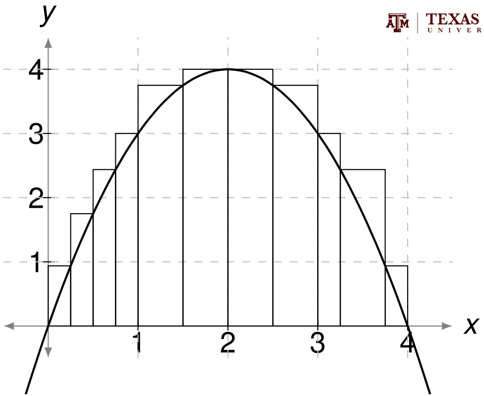

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(of 1) This comes from the fact that $m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\} \leq \sup\{f(x) : x \in [x_{k-1}, x_k]\} = M_k$.

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Definition

A bounded function f defined on the interval $[a, b]$ is **Riemann-integrable** if $U(f) = L(f)$. We write:

$$\int_a^b f = U(f) = L(f).$$

Theorem

A bounded function f is integrable on $[a, b]$ iff, for every $\varepsilon > 0$, there exists a partition P_ε of $[a, b]$ such that $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$.

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($\Leftarrow$) Let P_1 and P_2 be partitions of $[a, c]$ and $[c, b]$ respectively. Fix $\varepsilon > 0$, and let P_1, P_2 be partitions such that $U(f, P_1) - L(f, P_1) < \frac{\varepsilon}{2}$ and $U(f, P_2) - L(f, P_2) < \frac{\varepsilon}{2}$.

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Letting $\varepsilon \to 0$, we are done.

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\int_a^b (f+g) &\le U(f+g,P) \\
&\le U(f,P) + U(g,P) \\
&< L(f,P) + L(g,P) + \varepsilon \\
&\le \int_a^b f + \int_a^b g + \varepsilon.
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Theorem: $\int_a^b (f+g) = \int_a^b f + \int_a^b g$

Proof: With ε, P as before,

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—

Letting $\varepsilon \to 0$, we get the desired equality.

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Proof.

Exercise. For (iv), you may be able to use something from (i), (ii), and/or (iii) to prove the inequality. ∎

The Fundamental Theorem of Calculus

We now arrive at one of the apexes of analysis - the ability to connect derivatives and integrals together.

Theorem

(i) If $f : [a, b] \to \mathbb{R}$ is integrable, and $F : [a, b] \to \mathbb{R}$ satisfies $F'(x) = f(x)$ for all $x \in [a, b]$, then

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(ii) Let $g : [a, b] \to \mathbb{R}$ be integrable, and for $x \in [a, b]$, define

$$G(x) = \int_a^x g.$$

Then G is continuous on $[a, b]$. If g is continuous at some point $c \in [a, b]$, then G is differentiable at c and $G'(c) = g(c)$.

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Proof of (ii).

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Note f is bounded since it is integrable, so there is some M such that $|\int_x^c f| \leq M|x - c|$. So as we limit x to approach c, we get that $|G(x) - G(c)|$ goes to zero. ∎

Recall that FTC(ii) says in brief that, if a function g is integrable and $G(x) := \int_a^x g$, then $G'(c) = g(c)$.

Proof of (ii) (Continued).

Now we wish to calculate the derivative of G. We use the definition of the derivative:

$$\lim_{x \to c} \frac{G(x) - G(c)}{x - c} = \lim_{x \to c} \frac{\int_x^c g}{x - c}.$$

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The last equality is true since g is continuous at c. ∎

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We also learned about u-substitution, one of the main tools we use in solving integrals:

Theorem (Change of Variables)

Let $g : [a, b] \to \mathbb{R}$ be a continuously differentiable function, let $f : [c, d] \to \mathbb{R}$ be continuous, and suppose $g([a, b]) \subset [c, d]$ (notice the similarity to the set-up for Chain Rule).

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$$\int_{g(a)}^{g(b)} f(s)\,ds =: F(g(b)) = (F \circ g)(b) - (F \circ g)(a)$$

$$\overset{\text{FTC(i)}}{=} \int_a^b (F \circ g)'(x)\,dx = \int_a^b f(g(x))g'(x)\,dx.$$

∎

Logarithms and Exponentials

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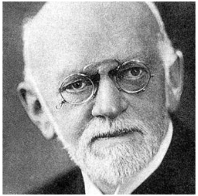
Does this converge to a rational or irrational number? This is one of the most popular types of problems from the twentieth century, thanks to David Hilbert.



John picked this picture because of how nerdy he looks

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Even here we struggle, because it appears that we have just substituted one irrational power for a worse one. But remember that the *exponential* function e^x can be defined using a very nice *Taylor series*... and there are lots of other tricks we have up our sleeve once a number comes into this form.

Theorem (Taylor Expansion of e^x)

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Calculators will often use formulas like this one to approximate these strange exponents.

Proposition (What function could this be?)

There exists a unique function $L : (0, \infty) \to \mathbb{R}$ such that

1. $L(1) = 0$.
2. *L is differentiable and* $L'(x) = \frac{1}{x}$.
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(5) can be done in a similar way. ∎

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Now compose both sides of this equation with E.

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Now compose both sides of this equation with E. Since E is continuous (by our previous proof), the limit can come outside E, so we get $\lim_{n\to\infty} e^{\ln\left(1+\frac{x}{n}\right)^n} = e^x$, as desired.

Improper Integrals

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Suppose $f : [a, b) \to \mathbb{R}$ is a function (not necessarily bounded) that is Riemann integrable on $[a, c]$ for all $c < b$. We define the **improper integral** of f on $[a, b]$ to be

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For example, $\int_0^1 \frac{1}{\sqrt{x}} = \lim_{b\to 0^+} \int_b^1 \frac{1}{\sqrt{x}} = \lim_{b\to 0^+} 2\sqrt{x}|_b^1 = \lim_{b\to 0^+}(2 - 2\sqrt{b}) = 2$.

The same is true for intervals of integration that are themselves unbounded, like $[a, \infty)$. These integrals *should* exist, but in order to extract as much information from them as possible we ask for them to be defined using a limit as well:

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Example: $\int_1^\infty \frac{1}{x^2} = \lim_{b\to\infty} \int_1^b \frac{1}{x^2} = \lim_{b\to\infty} -\frac{1}{x}\Big|_1^b = \lim_{b\to\infty}(1 - \frac{1}{b}) = 1.$

Using this definition we get lots of similar theorems to our integral theorems... and some theorems that might remind us of series:

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Let $f, g : [a, \infty) \to \mathbb{R}$ be functions that are Riemann integrable on $[a, b]$ for all $b > a$. Suppose further that $|f(x)| \leq g(x)$ for all $x \geq a$.
If $\int_a^\infty g$ converges, then $\int_a^\infty f$ converges too, and $|\int_a^\infty f| \leq \int_a^\infty g$.
If $\int_a^\infty f$ diverges, then $\int_a^\infty g$ diverges.

Let $f : [a, \infty) \to \mathbb{R}$ be integrable on $[a, b]$ for all $b > a$. Then for all $b > a$, $\int_b^\infty f$ converges iff $\int_a^\infty f$ converges, and $\int_a^\infty f = \int_a^b f + \int_b^\infty f$ (tail-ends of integrals).

Let (x_n) diverge to infinity. Then $\int_a^\infty f$ converges iff $\lim \int_a^{x_n} f$ exists, in which case $\int_a^\infty f = \lim_{n\to\infty} \int_a^{x_n} f$ (think “partial sums" of integrals).

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One piece connects these stories together: if an improper integral with an interval clustering toward infinity exists, its value can be thought of as a series of (proper) integrals! For example,

$$\int_0^\infty f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \cdots$$

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$$\int_{2\pi}^{\infty} \frac{\sin(x)}{x}\,dx = \int_{2\pi}^{4\pi} \frac{\sin(x)}{x}\,dx + \int_{4\pi}^{6\pi} \frac{\sin(x)}{x}\,dx + \cdots = \sum_{k=1}^{\infty} \int_{2k\pi}^{2(k+1)\pi} \frac{\sin(x)}{x}\,dx.$$

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$$\frac{1}{(2k+1)\pi} \leq \frac{1}{x} \leq \frac{1}{2k\pi} \Rightarrow \frac{\sin(x)}{(2k+1)\pi} \leq \frac{\sin(x)}{x} \leq \frac{\sin(x)}{2k\pi}.$$

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What's different about $\sin(x)$ on $[(2k+1)\pi, (2k+2)\pi]$? Then since multiplying by a negative number flips the signs, we should get

$$\frac{1}{(2k+2)\pi} \leq \frac{1}{x} \leq \frac{1}{(2k+1)\pi} \Rightarrow \frac{\sin(x)}{(2k+1)\pi} \leq \frac{\sin(x)}{x} \leq \frac{\sin(x)}{\pi(2k+2)}.$$

Note that terms like $\frac{\sin(x)}{(2k+1)\pi}$ are easy to integrate! (Why?)

$$\int_{2k\pi}^{(2k+1)\pi} \frac{\sin(x)}{(2k+1)\pi} \leq \int_{2k\pi}^{(2k+1)\pi} \frac{\sin(x)}{x} \leq \int_{2k\pi}^{(2k+1)\pi} \frac{\sin(x)}{2k\pi} \Rightarrow$$

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Added together: $0 \le \int_{2k\pi}^{2(k+1)\pi} \frac{\sin(x)}{x} \le \frac{1}{k(k+1)\pi}.$

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The **sinc function** $f(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ is a very important function for *partial differential equations*, which are used in biology, biochemistry, and physics research, along other applications.

Sequences of Functions

We return to Taylor polynomials: given a smooth function f, the Taylor polynomials P_n *approximate* the function f on its domain.

$$P_n(x) := f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

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The Taylor polynomials form a **sequence of functions**: the sequence $(P_1, P_2, P_3, \ldots)$ now has *functions* as its entries rather than real numbers. Historically, this machinery was developed to *approximate functions* with other more basic function types. (For example, polynomials can approximate the exponential function e^x.)

In fact, recall Taylor's Theorem:

Theorem (Taylor)

Let f possess at least $n+1$ derivatives on an open interval I and let $c \in I$. Let $R_n(x) = f(x) - P_n(x)$, where $P_n(x)$ is the nth Taylor polynomial centered at c. Then for each $x \in I$ there exists z between x and c such that

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Question: does the sequence of functions $f_n(x) := x^n$ converge pointwise to $f(x) = 0$ on the interval $[0, 1)$? **Yes.** Since $x < 1$, $x^n \to 0$ as $n \to \infty$. So $|f_n(x) - f(x)| \to 0$ as $n \to \infty$; this is the definition of pointwise convergence.

Definition

For each $n \in \mathbb{N}$, let $f_n : S \to \mathbb{R}$ be a function. Let $f : S \to \mathbb{R}$ be another function. Then we say f_n **converges pointwise** to f if, for every $x \in S$, we have $f(x) = \lim_{n\to\infty} f_n(x)$.

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Definition

Let f_n, f be as above. We say that f_n **converges uniformly** to f if, for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that, for $n > N$, $|f_n(x) - f(x)| < \varepsilon$ for all $x \in S$.

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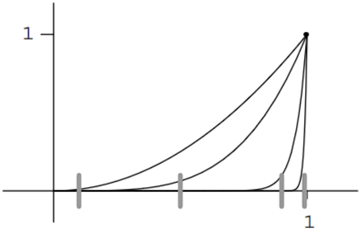

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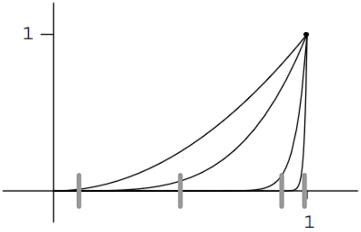

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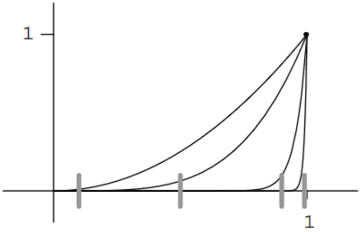

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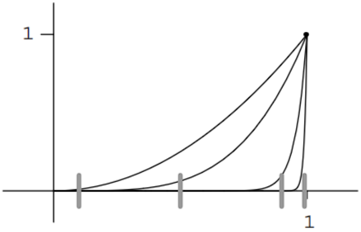

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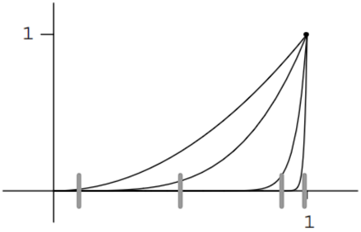

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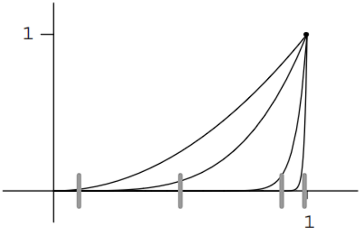

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