Solutions to Texas A&M's Real Analysis Qual Courses

Texas A&M Grad Students

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These are the solutions to the majority of the available past real analysis qualifying exams for Texas A&M. Incomplete/non-existent solutions are marked in red. If you find any errors/typos or have solutions to the unsolved questions, please email the current maintainer at jweeks03@tamu.edu. Credit for the founding of this document, the template, and the majority of these solutions goes to Kari Eifler. As a whole the solutions here are a collaborative effort with some solutions belonging to John Griffin, Wonhee Na, Jun Sur Park, Garrett Tresch, John Weeks, Zhiyuan Yang, and Byeongsu Yu, as well as those noted in the text. Kari Eifler and John Weeks are the main compilers of these solutions.

Contents

Note: different solution authors use differing notation for "is a (proper) subset of" ($\subset$,$\subseteq$) as well as for "the (closed) ball of radius r at a point x" ($B(r,x)$, $B(x,r)$); please be sure to interpret this notation contextually. Another common notation throughout is that of $[n] := \{1, 2, \ldots, n\}$.

1 January 2024

1. Let E be a Lebesgue measurable set of positive measure. Show that for any $0 < \alpha < 1$, there is an open interval I such that $m(E \cap I) > \alpha m(I)$.

Proof. One can directly prove this using Lebesgue differentiation theorem. Or one can use the definition of Lebesgue measure: Assume the statement is not true. Fix $\varepsilon > 0$. Let $\{U_i\}_{i\in I}$ be an countable family of open interval that covers E, such that $m(E) \geq \sum_{i\in I} m(U_i) - \varepsilon$. We have

$$m(E) \geq 1/\alpha \sum_{i\in I} m(U_i \cap E) - \varepsilon = 1/\alpha m(E) - \varepsilon,$$

where we used the fact that $\bigcup_{i\in I} U_i \cap E = E$. Let $\varepsilon \to 0$, we obtain $m(E) \geq 1/\alpha m(E)$, contradiction. $\square$

2. Let $(x_n)_{n=1}^{\infty}$ be a sequence in $[0, 1]$, and $(c_n)_{n=1}^{\infty}$ be a sequence of non-negative numbers such that $\sum_{n=1}^{\infty} c_n < \infty$. Show that the series

$$\sum_{n=1}^{\infty} \frac{c_n}{|x - x_n|^{1/2}}$$

converges for almost every $x \in [0, 1]$.

Proof. Since for every x the series $\sum_{n=1}^{\infty} \frac{c_n}{|x-x_n|^{1/2}}$ is positive, it either converges or tends to ∞. Therefore, it suffices to show that the integral

$$\int_0^1 \sum_{n=1}^{\infty} \frac{c_n}{|x - x_n|^{1/2}} dx$$

is finite. By Tonelli's theorem, we have

$$\sum_{n=1}^{\infty} \int_0^1 \frac{c_n}{|x - x_n|^{1/2}} dx = \sum_{n=1}^{\infty} 2c_n(\sqrt{x_n} + \sqrt{1 - x_n}) \leq 4 \sum_{n=1}^{\infty} c_n < \infty.$$

$\square$

3. Find the sum

$$\sum_{k=2}^{\infty} (-1)^k \sum_{n=2}^{\infty} \frac{1}{n^k}.$$

Proof. Note that we can not use Fubini's theorem as the double summation does not converges absolutely.

$$\begin{aligned}&\sum_{k=2}^{\infty}(-1)^k\sum_{n=2}^{\infty}\frac{1}{n^k}=\lim_{K\to\infty}\sum_{k=2}^{K}(-1)^k\sum_{n=2}^{\infty}\frac{1}{n^k}=\lim_{K\to\infty}\sum_{n=2}^{\infty}\sum_{k=2}^{K}(-1)^k\frac{1}{n^k}\\=&\lim_{K\to\infty}\sum_{n=2}^{\infty}\frac{1}{n^2}\frac{1-(-1/n)^{K-1}}{1+1/n}=\sum_{n=2}^{\infty}\frac{1}{n(n+1)}+\lim_{K\to\infty}\sum_{n=2}^{\infty}\frac{1}{n^2}\frac{(-1/n)^{K-1}}{1+1/n}\\=&\frac{1}{2}+\lim_{K\to\infty}\sum_{n=2}^{\infty}O\left(\frac{1}{2^{K-1}}\right)\frac{1}{n(n+1)}=\frac{1}{2},\end{aligned}$$

where we summed up the telescoping series $\sum_{n=2}^{\infty}\frac{1}{n(n+1)}$ in the last line. □

4. Let f be a Lebesgue measurable function on $[0,1]$ such that $f>0$ a.e. Suppose $(E_n)_{n=1}^{\infty}$ is a sequence of measurable sets with the property that

$$\int_{E_n} f dx \to 0.$$

Prove that $m(E_n)\to 0$.

Proof. Let $S_\delta=\{f\le\delta\}$ for $\delta>0$ and $\varepsilon>0$. Since $\lim_{\delta\to 0_+} m(S_\delta)=0$, we can pick a δ such that $m(S_\delta)<\varepsilon$. Now, since $\int_{E_n} f dx\to 0$, we can pick $N>0$ such that for all $n\ge N$,

$$\int_{E_n} f dx < \varepsilon\delta.$$

Now, we have

$$m(E_n\cap S_\delta^c)\le\frac{1}{\delta}\int_{E_n\cap E_\delta^c} f dx\le\frac{1}{\delta}\int_{E_n} f dx<\varepsilon.$$

Finally, $m(E_n)=m(E_n\cap S_\delta)+m(E_n\cap E_\delta^c)<m(S_\delta)+\varepsilon<2\varepsilon$. □

5. Recall that a point x is called isolated if $\{x\}$ is an open set. Show that a compact metric space with no isolated points is uncountable.

Proof. For any non-isolated $x\in X$, $\{x\}$ is nowhere dense. We then apply Baire category theorem. □

6. Recall that the graph of a function $f: X\to Y$ is the subset $\{(x,f(x)) : x\in X\}\subseteq X\times Y$.

(a) State the closed graph theorem.

(b) Give an example of a discontinuous function $f:\mathbb{R}\to\mathbb{R}$ whose graph is closed. Here $\mathbb{R}$ has standard topology.

(c) Give an example of a discontinuous linear function $f: X\to Y$, where X and Y are both normed spaces, whose graph is closed.

Proof. (a) A linear map between two Banach spaces is bounded if and only if its graph is closed.

(b)

$$f(x) = \begin{cases} 0, & x = 0, \\ \frac{1}{x}, & x \neq 0. \end{cases}$$

(c) Let $Y = (C[0,1], \|\cdot\|_\infty)$ with the uniform norm as usual, $X = (C[0,1], \|\cdot\|_1)$ with the L^1 norm, and $f : X \to Y$ the indentity map. Then f is unbounded as the L^∞-norm is not bounded above by the L^1-norm and f is closed since the L^∞-limit and L^1-limit of any sequence must be the same function (suppose they both exist.) □

7. Let $(X, \mathcal{M}, \mu)$ be a measure space with μ a probability measure. Show that $\|f\|_p$ is an increasing function of p for $0 < p \leq \infty$.

Proof. The case for infinity norm follows trivially. If $0 < p < q < \infty$, then by Hölder inequality

$$\|f\|_p = \|f^p \cdot 1\|_1^{1/p} \leq \|f^p\|_{q/p}^{1/p} \|1\|_{q/(q-p)}^{1/p} = \|f\|_q.$$

□

8. Let X and Y be reflexive Banach spaces such that Y^* is separable and there exists a continuous linear transformation T from X to Y with kernel $\{0\}$. Prove that X^* is separable.

Proof. As Y^* is separable,it suffices to show $T^*(Y^*)$ is dense in X^*. Suppose $\overline{T^*(Y^*)} \neq X^*$, take a nonzero $x^* \in X^* \setminus \overline{T^*(Y^*)}$ and pick a nonzero $x \in X^{**} = X$ such that $x(x^*) = 1$ and x vanishes on $\overline{T^*(Y^*)}$ (Hahn-Banach). Now,

$$0 = (x, T^*(y^*)) = y^*(Tx), \quad \forall y^* \in Y^*,$$

therefore $Tx = 0$ as Y^* separate points in Y. As $\ker T = \{0\}$, $x = 0$, contradiction. □

9. Let $\mathcal{P}$ be the space of real-valued polynomials, and $\mathcal{P}_n$ the subspace of polynomials of degree at most n. Fix $a \in \mathbb{R}$.

(a) Show that for every n, there exists a unique $g_n \in \mathcal{P}_n$ such that for all $f \in \mathcal{P}_n$,

$$f(a) = \int_0^1 f(x) g_n(x) dx.$$

(b) Show that there does not exist a Lebesgue integrable $h \in L^1([0,1], dx)$ such that for all $f \in \mathcal{P}$,

$$f(a) = \int_0^1 f(x) h(x) dx.$$

Proof. (a) Note that $\mathcal{P}_n$ is a finite dimensional vector space with inner product $\langle f, g \rangle = \int fg dx$, so it is a Hilbert space. Since any linear functional on finite dimensional space is bounded, the linear map $f \mapsto f(a)$ is bounded and the statement follows from Riesz representation theorem.

(b) Suppose there is such a h. Since $\mathcal{P}$ is dense in $C[0,1]$, we also have $f(a)=\int_0^1 f(x)h(x)dx$ for all $f\in C[0,1]$. For any $f\in C[0,1]$, $f\geq 0$, we have $\int_0^1 f(x)h(x)dx=f(a)\geq 0$, so $h(x)dx$ is a positive Borel measure and in particular $h\geq 0$ almost everywhere. Now let $f(x)=(x-a)^2$,

$$0=\int_0^1 (x-a)^2h(x)dx,$$

which implies $h=0$ a.e., a contradiction. □

10. For $f\in C[0,1]$, denote $\mathrm{co}(f)$ the smallest closed convex subset of $\mathbb{R}$ containing $\{f(x):0\leq x\leq 1\}$. Let Φ be a linear mapping from $C[0,1]$ to $\mathbb{R}$ such that $\Phi(f)\in\mathrm{co}(f)$ for each f. Prove that

$$\lim_{n\to\infty}\Phi\left(\frac{n^2}{(nx-1)^2+n^2}\right)=\Phi\left(\frac{1}{1+x^2}\right).$$

Proof. First, note that as $[0,1]$ is connected and compact, $f([0,1])$ is also connected and compact and thus is a closed interval. In particular, $\mathrm{co}(f)$ is nothing but the image $f([0,1])$.

$$\begin{aligned}\Phi\left(\frac{n^2}{(nx-1)^2+n^2}\right)-\Phi\left(\frac{1}{1+x^2}\right)&=\Phi\left(\frac{1}{1+(x-\frac{1}{n})^2}-\frac{1}{1+x^2}\right)\\ =&\Phi\left(\frac{\frac{2x}{n}-\frac{1}{n^2}}{(1+(x-\frac{1}{n})^2)(1+x^2)}\right)=\frac{1}{n}\Phi\left(\frac{2x-\frac{1}{n}}{(1+(x-\frac{1}{n})^2)(1+x^2)}\right).\end{aligned}$$

Since $\frac{2x-\frac{1}{n}}{(1+(x-\frac{1}{n})^2)(1+x^2)}\in[-1,2]$ for all x and n, we have $|\Phi\left(\frac{2x-\frac{1}{n}}{(1+(x-\frac{1}{n})^2)(1+x^2)}\right)|\leq 2$ for all n. In particular, $\Phi(\frac{n^2}{(nx-1)^2+n^2})-\Phi(\frac{1}{1+x^2})\to 0$ as $n\to\infty$. □

2 August 2023

1. Let $f:(0,1)\to\mathbb{R}$ be a Lebesgue integrable function. For any $x\in(0,1)$ define $g(x)=\int_x^1\frac{f(t)}{t}d\lambda(t)$. Prove that $g:(0,1)\to\mathbb{R}$ is integrable and $\int_0^1 fd\lambda=\int_0^1 gd\lambda$.

Proof. Since $\displaystyle\int_{(0,1)\times(0,1)}\mathbb{1}_{t\geq x}\cdot\frac{|f(t)|}{t}d\lambda(t)d\lambda(x)=\int_0^1|f(t)|d\lambda(t)=\|f\|_1$, we can apply Fubini. □

2.

(1) Let $(X,\mathcal{M},\mu)$ be a measure space and let $(f_n)_{n\geq1}$ be a sequence of measurable functions on X. Define what it means that $(f_n)_{n\geq1}$ converges in measure to a measurable function g.

(2) Give an example of a sequence of measurable functions that converges pointwise but not in measure.

(3) Let $(X,\mathcal{M},\mu)$ be a finite measure space and let $(f_n)_{n\geq1}$ be a sequence of measurable functions on X that converges pointside to g. Show that $(f_n)_{n\geq1}$ converges in measure to g.

Proof. (2) Consider $\mathbb{R}$ with the Lebesgue measure. Take $f_n(x) := \mathbb{1}_{x\geq n}$.

(3) $\lim_{n\to\infty} f_n(x) = g(x)$ means that $\forall\varepsilon > 0, \exists N \geq 1, \forall n \geq N, |f_n(x) - g(x)| < \varepsilon$. Therefore, the set of converging points of $(f_n)_{n\geq 1}$ is

$$X = \bigcap_{\varepsilon} \bigcup_{N\geq 1} \bigcap_{n\geq N} \{|f_n - g| < \varepsilon\}.$$

In particular, for all $\varepsilon > 0$, $\bigcup_{N\geq 1} \bigcap_{n\geq N} \{|f_n - g| < \varepsilon\} = X$. Taking the complement, one get

$$\mu(\bigcap_{N\geq 1} \bigcup_{n\geq N} \{|f_n - g| \geq \varepsilon\}) = 0.$$

Since $\bigcup_{n\geq N}\{|f_n - g| \geq \varepsilon\}$ is decreasing for N and μ is finite, we have $\lim_{N\to\infty} \mu(\bigcup_{n\geq N}\{|f_n - g| \geq \varepsilon\}) = \mu(\bigcap_{N\geq 1} \bigcup_{n\geq N}\{|f_n - g| \geq \varepsilon\}) = 0$. Therefore $\mu(\{|f_N - g| \geq \varepsilon\}) \leq \mu(\bigcup_{n\geq N}\{|f_n - g| \geq \varepsilon\}) \to 0$. □

3.

(1) Let $\{X_i\}_{i\in I}$ be a collection of topological space. Define what is the product topology on the Cartesian product $\prod_{i\in I} X_i$.

(2) Prove that a compact metric space is separable.

(3) Show that every compact metric space is homeomorphic to a closed subset of $[0, 1]^{\mathbb{N}}$ (equipped as usual with the product topology).

Proof. (2) For each $n \geq 1$, we can pick a finite set F_n, such that $\bigcup_{x\in F_n} B_{1/n}(x)$ is the whole space. Now $\bigcap_{n\geq 1} F_n$ is a countable dense subset, as for any ball $B_r(x_0)$, we can pick n such that $n < 1/(2r)$ and a $y \in F_n$ such that $x_0 \in B_{1/n}(y)$ which then implies $y \in B_r(x_0)$.

(3) WLOG assume $\mathrm{diam}(X) \leq 1$. Let $(x_n)_{n\geq 1}$ be an arrangement of a countable dense subset of X. Consider the map $i : X \to [0, 1]^{\mathbb{N}}$, $i(x) = (d(x, x_n))_{n\geq 1}$. i is injective: we have $x = \lim_k x_{n_k}$ for some sequence $(n_k)_{k\geq 1}$, and if $d(x, x_n) = d(y, x_n)$ for all n, then we also have $y = \lim_k x_{n_k} = x$. i is continuous by the property of the product topology as each of its coordinate $d(\cdot, x_n)$ is continuous. Therefore i is injective continuous function between compact Hausdorff spaces, which must be a homeomorphism onto its image (which is compact). □

4. In this problem, recall the duality $L_1(\mathbb{R})^* = L_\infty(\mathbb{R})$. Let $S = \{f \in L_\infty(\mathbb{R}) : \lambda(\{x : f(x) > \frac{1}{1+e^{-|x|}}\} = 0)\}$.

(1) State the Banach-Alaoglu theorem.

(2) Show that S is a weak*-compact subset of $L_\infty(\mathbb{R})$.

(3) Is S a norm-compact subset of $L_\infty(\mathbb{R})$?

Proof. (2) Since S is contained in the unit ball of $L_\infty(\mathbb{R})$ which is weak*-compact, it suffices to show that S is weak*-closed. For this we simply note that

$$f \in S \iff \forall g \in L_1(\mathbb{R}), g \geq 0 \ a.e., \int g(\frac{1}{1+e^{-|x|}} - f)d\lambda \geq 0$$

which is a property preserved under weak*-limit.

(3) No. Consider $f_n = \frac{1}{2}\mathbb{1}_{x\geq n}$, then $\|f_n - f_m\|_\infty = \frac{1}{2}$ for $n \neq m$, hence $(f_n)_{n\geq 1}$ has no converging subsequence. □

5. Let $k \in C([0,1]\times[0,1])$ and $f \in C([0,1])$ be real-valued continuous functions. Define the function $T(f) : [0,1] \to \mathbb{R}$:

$$T(f)(x) = \int_0^1 k(x,y)f(y)dy, \quad x \in [0,1].$$

(1) Show that $T(C[0,1]) \in C([0,1])$.

(2) Show that T maps bounded sets to subsets of compact sets.

Proof. (1) $|T(f)(x) - T(f)(x')| \geq \int_0^1 |k(x,y) - k(x',y)||f(y)|dy$. Apply the uniform continuity of k.
(2) Use Arzelà-Ascoli Theorem. □

6. Let $f : (0,\infty) \to \mathbb{R}$ be continuous and such that for all $x > 0$ the sequence $(f(nx))_{n\geq 1}$ converges to 0. Show that $\lim_{x\to\infty} f(x)$.

Proof. Writing $\forall x, \lim_{n\to\infty} f(nx) = 0$ in terms of set, we have

$$\mathbb{R} = \bigcap_{\varepsilon}\bigcup_{N\geq 1}\bigcap_{n\geq N}\{x : |f(xn)| \leq \varepsilon\}.$$

So for each fixed $\varepsilon > 0$, $\bigcup_{N\geq 1}\bigcap_{n\geq N}\{x : |f(xn)| \leq \varepsilon\} = \mathbb{R}$. Note that for each N, $E_N = \bigcap_{n\geq N}\{x : |f(xn)| \leq \varepsilon\}$ is closed, and $(E_N)_{N\geq 1}$ is increasing.

Claim: There exists N_0, $a, b \in \mathbb{R}, a < b$ such that $[a,b] \subseteq E_{N_0}$.

To see this one can directly apply Baire Category theorem, or one can prove this directly: Assume the claim is not true. Since E_N^c is open, it is a union of countable disjoint open intervals. For $N = 1$, take $[a_1, b_1] \subset (a_1', b_1') \subseteq E_1^c$. For $N \neq 2$, as (a_1, b_1) is not contained in E_2, $(a_1, b_1) \cap E_2^c$ is nonempty open subset. One can then take $[a_2, b_2] \subset (a_2', b_2') \subseteq E_2^c \cap (a_1, b_1)$. Continuing this process, we obtain a sequence of decreasing closed interval $[a_N, b_N] \subset E_N^c$. By the completeness of $\mathbb{R}$, $\cap_N [a_N, b_N]$ must be empty, contradicting to the fact that $\cap_N E_N^c = \emptyset$.

Now, let $F_\varepsilon = \{x : |f(x)| < \varepsilon\}$. We have $nE_N \subseteq F_\varepsilon$ for all $n \geq N$. In particular, as $[a,b] \subseteq E_{N_0}$, $[na, nb] \subseteq F_\varepsilon$ for all $n \geq N_0$. One can then easily check that there exists a $n_0 \geq N_0$ such that $[n_0 a, \infty) \subseteq \bigcup_{n\geq n_0}[na, nb] \subseteq F_\varepsilon$. (Indeed, one simply need to take n_0 to be the numerator of a rational number $\frac{n_0}{m_0} < \frac{a}{b}$.) □

7.

(1) State the Hahn-Banach theorem (for sublinear and linear functionals on a real vector space).

For the next two questions, X is a real normed vector space and we denote by X^* the dual space.

(2) Let C be a convex open set in X that contains 0 and define for all $x \in X$, $p_C(x) = \inf\{\alpha > 0 : x \in \alpha C\}$. Show that p_C is a sublinear functional and $C = \{x \in X : p_C(x) < 1\}$.

(3) Let C be an open, convex, nonempty subset of X and $x_0 \notin C$. Show that there is $x^* \in X^*$ such that $x^*(z) < x^*(x_0)$ for all $z \in C$.

Proof. (1) Sublinear: $p(x+y) \geq p(x) + p(y)$, $p(tx) = tp(x)$ for all $x, y \in X$, $t \geq 0$. HB thm: If f is a linear functional on a linear subspace Y of X such that $f \leq p$ on Y, then there exists linear extension $\tilde{f}$ of f on X such that $\tilde{f} \leq p$ on X.

(2) $p_C(tx) = tp_C(x)$ is easy to show. If $p_C(x) = \alpha$, $p_C(y) = \beta$, then for any $\varepsilon > 0$, $x \in (\alpha + \varepsilon)C$ and $y \in (\beta + \varepsilon)C$. Therefore $x + y \in (\alpha + \beta + 2\varepsilon)C$ and hence $p_C(x+y) \geq \alpha + \beta$. It is clear that $\{x \in X : p_C(x) < 1\} \subseteq C$. To see the other direction, take $x \in C$. As C is open, there exists a $\delta > 0$ such that $(1+\delta)x$ is still in C, therefore $p_C(x) \leq 1/(1+\delta) < 1$.

(3) WLOG assume $0 \in C$. Consider the linear functional $f : \mathbb{R}x_0 \to \mathbb{R}$, $f(tx_0) = tp_C(x_0)$. Then $f \leq p_C$ on $\mathbb{R}x_0$. Let $\tilde{f}$ be a linear extension of f such that $\tilde{f} \leq p_C$. Then we have

$$\forall z \in C, \tilde{f}(z) \leq p_C(z) < 1 \leq p_C(x_0).$$

Finally, we need to check that $\tilde{f} \in X^*$. Let $B_\delta(0) \subseteq C \cap (-C)$ be an open ball centered at $0 \in X$, then by the previous inequality $-1 < \tilde{f}(z) < 1$ for all $z \in B_\delta(0)$. In particular, $\tilde{f}(B_1(0)) \subseteq (1/\delta, 1/\delta)$ which implies the boundedness of $\tilde{f}$. ☐

8.

(1) Given a Banach space $(X, \|\cdot\|)$ and a sequence $(x_n)_{n\geq 1} \in X$. What does it mean that $(x_n)_{n\geq 1}$ converges weakly to $x \in X$?

(2) Let K be a compact Hausdorff space. Show that a sequence $(f_n)_{n\geq 1}$ in $C(K)$ converges weakly iff $(f_n)_{n\geq 1}$ is bounded and converges pointwise.

Proof. See Problem 7, January 2019. ☐

9. Let $1 < p < q < r, \infty$ and $f \in [0,1] \to \mathbb{R}$ be measurable. Show that

$$\|f\|_q \leq \|f\|_p^s \|f\|_r^{1-s}$$

where $s = \frac{\frac{1}{q} - \frac{1}{r}}{\frac{1}{p} - \frac{1}{r}}$.

Proof. Apply the generalized Hölder inequality $\|gh\|_{r'} \leq \|g\|_{p'} \|h\|_{q'}$ by taking $\dfrac{1}{r'} = \dfrac{1}{p'} + \dfrac{1}{q'}$ with $g = f^s$, $h = f^{1-s}$, $p' = p/s$, $q' = r/(1-s)$ and $r' = q$. ☐

10.

(1) Give the definition of an orthonormal basis in a Hilbert space.

(2) Show that if a Hilbert space has a countably infinite orthonormal basis then every infinite orthonormal basis is countable.

Proof. (1) $\{e_n\}_{n\in I}$ family of unit vector such that $\langle e_n, e_m\rangle = \delta_{m,n}$ and $\text{span}\{e_n\}_{n\in I}$ is dense in $\mathcal{H}$. (2) If $\mathcal{H}$ has an countable orthonormal basis, then it is separable. But if $\mathcal{H}$ has an uncountable orthonormal basis, then it is not separable: let $\{e_n\}_{e\in I}$ be an uncountable orthonormal basis, then $\{B_{1/3}(e_n)\}_{n\in I}$ is an uncountable family of disjoint open balls. □

3 January 2023

1. Show that there exists a constant $c > 0$ (and give its value) so that for every measurable function $f : \mathbb{R} \to [0, \infty)$ we have

$$\int_{\mathbb{R}} f^4 d\lambda = c \int_{[0,\infty)} t^3 \lambda(\{f \geq t\}) d\lambda(t).$$

Proof. By Tonelli theorem,

$$\int_{[0,\infty)} 4t^3 \lambda(\{f \geq t\}) d\lambda(t) = \int_{\mathbb{R}\times[0,\infty)} 4t^3 \cdot \mathbb{1}_{f(s)\geq t} d\lambda(s) d\lambda(t)$$
$$= \int_{\mathbb{R}} f(s)^4 d\lambda(s) = \int_{\mathbb{R}} f^4 d\lambda.$$

□

2. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of integrable functions from X to $\mathbb{R}$ such that $\lim_{n\to\infty} \int_X |f_n - f| d\mu$ for some integrable function $f : X \to \mathbb{R}$. Show that for all $\varepsilon > 0$ there is $A \in \mathcal{M}$ satisfying $\mu(A) < \infty$ and for all $n \geq 1$,

$$\int_{X\setminus A} |f_n| d\mu < \varepsilon.$$

Proof. Let us fix $\varepsilon > 0$. Since $f \cdot \mathbb{1}_{f\geq m} \to 0$ a.e. when $m \to \infty$, we have $\lim_m \int_{X\setminus\{f\geq m\}} |f| d\mu = 0$. Therefore, we can take $B = \{f \geq m_0\}$ for a large enough m_0, such that $\int_{X\setminus B} |f| d\mu < \varepsilon/2$. As f is integrable, $\mu(B) \leq \|f\|_1/m_0 < \infty$. Also as $\lim_{n\to\infty} \int_X |f_n - f| d\mu$, we can take $N \geq 1$ such that for all $n \geq N$, $\int_{X\setminus B} |f_n - f| d\mu \leq \int_X |f_n - f| d\mu < \varepsilon/2$. In particular,

$$\int_{X\setminus B} |f_n| d\mu \leq \int_{X\setminus B} |f_n - f| d\mu + \int_{X\setminus B} |f| d\mu < \varepsilon, \quad \forall n \geq N.$$

Now, for each f_n with $n < N$, just like for f, we can pick a measurable subset B_n with finite measure, and $\int_{X\setminus B_n} |f_n| d\mu < \varepsilon$. Finally, $A := B \cup \bigcup_{n<N} B_n$ satisfies the statement.

□

3. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions from $[0,1]$ to $\mathbb{R}$.

(1) Show that if $\lim_{n\to\infty} |f_n - f| d\lambda = 0$ for some integrable f, then $(f_n)_{n\in\mathbb{N}}$ converges in λ-measure to f.

(2) Show that if $(f_n)_{n\in\mathbb{N}}$ converges λ-almost everywhere towards a measurable function $f : [0,1] \to \mathbb{R}$, then $(f_n)_{n\in\mathbb{N}}$ converges in λ-measure to f.

(3) Does the conclusion in assertion (2) still hold if the functions are defined on $\mathbb{R}$ instead?

Proof. (1) For all $\varepsilon > 0$, $\lambda(\{|f_n - f| > \varepsilon\}) \leq \int_{\mathbb{R}} |f_n - f| d\lambda/\varepsilon \to 0$ as $n \to \infty$. For (2), (3), see Problem 2 from August 2023. □

4. Recall that a collection $\mathcal{F}$ of measurable functions from $[0,1]$ to $\mathbb{R}$ is said to be uniformly integrable if

$$\lim_{\lambda(A)\to 0} \sup_{f\in\mathcal{F}} \int_A |f| d\lambda = 0.$$

(1) Given a non-negative $g \in L_1([0,1])$, show that $\mathcal{F}_g := \{f \in L_1([0,1]) : |f| \leq g\}$ is uniformly integrable.

(2) Show that the closed unit ball of $L_2([0,1])$ is a uniformly integrable subset of $L_1([0,1])$.

Proof. (1) For all measurable A, $f \in \mathcal{F}_g$, $\int_A |f| d\lambda \leq \int_A g d\lambda$, hence

$$\lim_{\lambda(A)\to 0} \sup_{f\in\mathcal{F}_g} \int_A |f| d\lambda = \lim_{\lambda(A)\to 0} \int_A g d\lambda = 0,$$

where the last step is due to dominant convergence theorem.

(2) For each f with $\|f\|_2 \leq 1$, one have by Cauch-Schwarz inequality

$$\int_A |f| d\lambda \leq \|f\|_2 (\lambda(A))^{1/2} \leq (\lambda(A))^{1/2},$$

which converges uniformly to 0 as $\lambda(A) \to 0$. □

5.

(1) Show that a compact metric space is separable.

(2) Prove or disprove that the unit ball of ℓ_∞ equipped with the norm topology is separable.

(3) Prove or disprove that the unit ball of ℓ_∞ equipped with the weak* topology is separable.

Proof. (1) See Problem 3 from August 2023. (2) FALSE, for any subset $S \subseteq \mathbb{N}$, let $f_S := (\mathbb{1}_{n\in S})_n$. Then $(f_S)_S$ is an uncountable family such that $\|f_S - f_{S'}\| = 1$ for all $S \neq S'$, hence $(B_{1/3}(f_S))_S$ is an disjoint uncountable family of open balls, which forces the unit ball of ℓ_∞ to be non-separable in norm.

(3) TRUE, by Banach-Alaoglu, the unit ball of ℓ_∞ is weak*-compact. Also, since the predual ℓ_1 of ℓ_∞ is separable, $(\ell_\infty)_1$ has a countable family of separating seminorms in the weak* topology, and hence has a translation-invariant metric. In particular, the unit ball of ℓ_∞ with weak* topology is a compact metric space and thus is separable. See Ch. 5, Exercise 50 from Folland. Indeed, let $(s_n)_{n\geq 1}$ be a countable dense subset of $(\ell_1)_1$, and define the metric on $(\ell_\infty)_1$:

$$d(f,g) := \sum_{n\geq 1} \frac{1}{2^n}|(s_n, f-g)|.$$

Claim: the topology defined by d coincide with the weak* topology on $(\ell_\infty)_1$.

First, assume that $d(f_m, f) \to 0$ when $m \to 0$ with $f, f_m \in (\ell_\infty)_1$. For every $s \in (\ell_1)_1$ and $\varphi > 0$, we can choose a s_n such that $\|s_n - s\|_1 \leq \varepsilon/4$. Choose also a $M > 0$, such that for all $m \geq M$, $d(f_m, f) \leq \varepsilon 2^{-n-1}$. We have then $|(s_n, f_m - f)| \leq 2^n d(f_m, f) \leq \varepsilon/2$. Therefore,

$$|(s, f_m - f)| \leq |(s_n - s, f_m - f)| + \varepsilon/2 \leq 2\|s_n - s\|_1 + \varepsilon/2 = \varepsilon,$$

which implies that $f_m \to f$ in the weak* topology.

On the other hand, assume that $f_m \to f$ in the weak* topology. For each $\varepsilon > 0$, choose a $N > 0$ such that $\frac{1}{2^{N-2}} \leq \varepsilon$. For each $i \leq N$, as $\lim_{m\to\infty}(s_i, f_m - f) = 0$, we can choose a $M > 0$ such that $|(s_i, f_m - f)| \leq \varepsilon/2$ for all $m \geq M$ and $i \leq N$. Now, for all $m \geq M$,

$$d(f_m, f) = \sum_{n=1}^{N} \frac{1}{2^n}|(s_n, f_m - f)| + \sum_{n\geq N+1} \frac{1}{2^n}|(s_n, f_m - f)| \leq \sum_{n=1}^{N} \frac{1}{2^n}\varepsilon/2 + \frac{1}{2^{N-1}} \leq \varepsilon.$$

□

6. For $f \in C[0,1]$, let

$$\|f\|_L = |f(0)| + \sup_{0\leq x<y\leq 1} \frac{|f(y) - f(x)|}{y - x}$$

(1) Show that $\{f \in C[0,1] : \|f\|_L \leq 1\}$ is compact in $C[0,1]$.

(2) Is the set $\{f \in C[0,1] : \|f\|_L < \infty\}$ dense in $C[0,1]$ or not?

Proof. First, notice that for all $f \in C[0,1]$, $x \in [0,1]$,

$$|f(x)| \leq |f(0)| + |x - 0|\frac{|f(x) - f(0)|}{|x - 0|} \leq 2\|f\|_L.$$

And for $x \neq y$, we have furthermore $|f(x) - f(y)| \leq |x - y|\|f\|_L$.

(1) Let $S = \{f \in C[0,1] : \|f\|_L \leq 1\}$. Then S is uniformly bounded by 2 and uniformly equicontinuous. Hence by Arzelà-Ascoli theorem S is precompact. It remains to show that S is closed. But since for each $0 \leq x < y \leq 1$, the map $i_{x,y} : C[0,1] \to \mathbb{R}$, $i_{x,y}(f) := |f(0)| + \frac{|f(y)-f(x)|}{y-x}$ is norm-continuous, we have

$$S = \bigcap_{0\leq x<y\leq 1} i_{x,y}^{-1}((-\infty, 1]),$$

which is closed.

(2) It is dense, as the set contains the set of all polynomials which is already dense in $C[0,1]$. $\square$

7. Suppose X is a real Banach space and $Y \subseteq X$ is a proper subspace. Show that the following are equivalent:

(1) For every $z \in X$ such that $z \notin Y$, there exists a bounded linear functional ϕ on X such that $\phi(z) = 1$ and, for all $y \in Y$, $\phi(y) = 0$.

(2) Y is closed in X.

Proof. (1) $\implies$ (2): For each $z \in X \backslash Y$, pick a $\phi_z \in X^*$ such that $\phi_z(z) = 1$ and $\phi_z = 0$ on Y. Then $Y = \bigcap_{z \in X \backslash Y} \phi_z^{-1}(\{0\})$ which is closed.

(2) $\implies$ (1): Use Hahn-Banach theorem. $\square$

8.

(1) Let X be a normed vector space and Y be a subspace of X. Show that if Y has non-empty interior then $Y = X$.

(2) Let X be a banach space and T be a bounded operator on X. Show that if for all $x \in X$, there exists $n \in \mathbb{N}$ such that $T^n(x) = 0$, then there exists $d \in \mathbb{N}$ such that for all $x \in X$, $T^d(x) = 0$.

Proof. (1) Since Y is a subspace, if Y has non-empty interior, then Y contains a neighborhood of 0. In particular, Y contains an open ball of X and thus must contains X.

(2) Consider $S_n = \{x \in X : T^n(x) = 0\}$ then $X = \cup_{n \geq 1} S_n$. Apply Baire category theorem and (1). $\square$

9. Let $(X, \| \cdot \|)$ be a normed vector space. A sequence $(x_n)_{n \in \mathbb{N}}$ in X is said to be weakly Cauchy if for all $x^* \in X^*$, $(x^*(x_n))_{n \in \mathbb{N}}$ is a Cauchy sequence.

(1) Show that a weakly Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ in X is bounded.

(2) Show that for every weakly Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ in X, there exists $x^{**} \in X^{**}$ such that $(x_n)_{n \in \mathbb{N}}$ weak*-converges to x^{**} and $\|x^{**}\| \leq \liminf_{n \to \infty} \|x_n\|$.

Proof. (1) Banach-Steinhaus. (2) Consider the linear map $f : X^* \to \mathbb{R}$, $f(x^*) = \lim_{n \to \infty} x^*(x_n)$. Then by (1), f is bounded, hence $f \in X^{**}$ and by definition x_n weak*-converges to f. Finally, the norm bound follows from the inequality

$$|f(x^*)| = |\lim_{n \to \infty} x^*(x_n)| = \lim_{n \to \infty} |x^*(x_n)| = \liminf_{n \to \infty} |x^*(x_n)| \leq \liminf_{n \to \infty} \|x^*\| \|x_n\|, \quad \forall x^* \in X^*$$

$\square$

10. Let $(g_n)_{n \in \mathbb{N}}$ be a sequence of non-negative continuous functions on $[0,1]$ such that for each $k \in \mathbb{N} \cup \{0\}$, the limit $\lim_{n \to \infty} \int_0^1 t^k g_n(t) d\lambda(t)$ exists. Show that there exists a unique finite positive Radon measure μ on $[0,1]$ such that for all continuous functions on $[0,1]$, $\int_0^1 f d\mu = \lim_{n \to \infty} \int_0^1 f(t) g_n(t) d\lambda(t)$.

Proof. Let $M \geq 0$ be an upper bound of $\int_0^1 g_n(t)d\lambda(t)$. The linear map T : polynomials $\to \mathbb{R}$, $T(t^k) := \lim_{n\to\infty} \int_0^1 t^k g_n(t)d\lambda(t)$ is bounded by M. We denote the bounded extension of T to $C[0,1]$ again by T. By Riesz representation theorem, T must coincide with some finite positive Borel measure μ (as $[0,1]$ is locally compact Hausdorff, every finite Borel measure is Radon). □

4 August 2022

Problem 1. *Let $f : (0,1) \to \mathbb{R}$.*

(a) Give the definition of absolute continuity of f.

(b) Show that if $E \subset (0,1)$ has Lebesgue measure 0 and f is monotone and absolutely continuous, then $f(E)$ has Lebesgue measure zero.

Proof. (a) Check Folland.

(b) (WLOG let f be monotone increasing.) Let $U_n = \sqcup_{j=1}^{\infty}(a_j^n, b_j^n)$ be a disjoint union of open intervals such that $m(U_n) < 1/n$ and $E \subset U_n$. Then since f is absolutely continuous, for all $k \in \mathbb{N}$

$$\sum_{j=1}^{k} f(b_j^n) - f(a_j^n) < \varepsilon,$$

so by taking a limit in k $\sum_{j=1}^{\infty} f(b_j^n) - f(a_j^n) \leq \varepsilon$. Note then (since f is monotone) that $m(f(U_n)) \leq \varepsilon$ as well. Since $f(E) \subset f(U_n)$, $m(f(E)) = 0$. □

Problem 2. *Suppose X is a compact Hausdorff space and $f : X \to \mathbb{R}$ is continuous. Let $\varepsilon > 0$. Show the existence of an open set $U \subset X$ and a continuous function $g : X \to \mathbb{R}$ such that $f^{-1}(\{0\}) \subset U$, $g(U) = \{0\}$, and $\|g - f\|_u < \varepsilon$, where the norm is the uniform norm.*

Proof. Let $U := f^{-1}((-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}))$ (note $f^{-1}(\{0\}) \subset U$) and $V := f^{-1}((-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}))$. Then $\overline{U} \cap V^c = \varnothing$, and by Urysohn there is some $h \in C(X)$ such that $h \equiv 1$ on V^c and $h \equiv 0$ on U.

Define $g = fh$. Then g is continuous and $g(U) = \{0\}$. Also $fh|_{V^c} = f|_{V^c}$ and $|(fh)(x)| \leq |f(x)|$ for all $x \in V$. Hence

$$\|fh - f\|_\infty < 2(\frac{\varepsilon}{2}) = \varepsilon.$$

□

Problem 3. *Let X be a complete metric space that is uncountable but has a countable dense subset D. Suppose $f : X \to X$ satisfies $f(X\backslash D) \subset D$ and $f(D) \subset X\backslash D$. Show that f cannot be everywhere continuous on X.*

Proof. This is similar to August 2021 #9, but only the first solution can be adapted. WLOG X is connected; otherwise we can consider an uncountable connected component of X, which would itself be a complete metric space. (If all connected components of X are countable, then since each component must contain at least one element of D (since D is dense in X) it must follow that X is countable.) Let $D = (d_i)_1^\infty$, and note $\bigcup_i f^{-1}(\{d_i\}) = X\backslash D$. If f is everywhere continuous, each

$f^{-1}(\{d_i\})$ is closed (since metric spaces are T_1) and nowhere dense (any open set in $f^{-1}(\{d_i\})$ must contain an element in D). So $X = \bigcup_i \{d_i\} \cup \bigcup_i f^{-1}(\{d_i\})$ is a countable union of nowhere dense sets (since $\{d_i\}$ is not open; otherwise X is not connected), and Baire Category Theorem yields a contradiction. Our only assumption was that f was everywhere continuous, so this must be untrue. □

Problem 4. *Either give an example of a σ-finite measure space $(X, \mathcal{M}, \mu)$ with an uncountable family $(A_\lambda)_{\lambda\in\Lambda}$ of pairwise disjoint measurable sets $A_\lambda \in \mathcal{M}$, each with $\mu(A_\lambda) > 0$, or prove that such an example cannot exist.*

Proof. By #1 in January 2010, we should believe this cannot exist, and that proof can be generalized. Let $X = \bigcup_{i=1}^\infty X_i$ where $\mu(X_i) < \infty$ for all i. Let $(A_i^j)_j$ be pairwise disjoint in X_i; then $\{j : m(A_i^j) > 1/n\}$ is finite since $\mu(X_i) < \infty$, so $\{j : m(A_i^j) > 0\} = \bigcup_n \{j : m(A_i^j) > 1/n\}$ is countable. Now let A_i be disjoint in X; then $\{i : m(A_i \cap X_j) > 0\}$ is countable by the above, so $\{i : m(A_i) > 0\} = \bigcup_j \{i : m(A_i \cap X_j) > 0\}$ is countable. □

Problem 5. *Let $\mathcal{V} \subset C[0,1]$ be the linear span of the polynomials $\{x^{2n} : n \in \mathbb{N}, n > 0\}$. For which values of p, $1 \le p \le +\infty$, is $\mathcal{V}$ dense in $L^p([0,1])$, (defined using Lebesgue measure on $[0,1]$, of course)? Prove that your answer is correct.*

Proof. By Stone-Weierstrass, $\mathcal{V}$ is uniformly dense in $\{f \in C[0,1] : f(0) = 0\}$. Now uniform convergence implies L^p-convergence for all p, and it is easy to find continuous functions (f_n) that converge to 1 in L^p for all $1 \le p < \infty$ (take $f_n \equiv 1$ on $[1/n, 1]$ and $f_n(x) = nx$ on $[0, 1/n]$). Note $C[0,1]$ is dense in $L^p[0,1]$ for $1 \le p < \infty$, so in fact $C[0,1]$ is dense in $L^p[0,1]$ for all $1 \le p < \infty$.

However, if $\text{span}\{x^{2n} : n > 0\}$ were dense in $L^\infty[0,1]$, then by taking the $(\mathbb{Q} + \mathbb{Q}i)$-span of $\{x^{2n} : n > 0\}$, which is uniformly dense (and hence L^p-dense) in the complex span, it would follow that $L^\infty[0,1]$ is separable. But $L^\infty[0,1]$ is not separable, as $\{1_{[r,r']}\}_{r<r'\in[0,1]}$ is an uncountable collection of $L^\infty[0,1]$ functions each of distance 1 away from each other (see August 2017 #4a). So any dense collection of $L^\infty[0,1]$ must contain an element of the 1/2-ball around each of these functions (which are disjoint) and hence must be uncountable. □

Problem 6. *Let $T : X \to Y$ be a bounded linear operator between Banach spaces. Let X^* and Y^*, respectively, denote the dual spaces consisting of bounded linear functionals of X and Y. Let $T^* : Y^* \to X^*$ by defined by $(T^*\phi)(x) = \phi(Tx)$ for $\phi \in Y^*$ and $x \in X$. (You may assume and need not prove that T^* is well defined.)*

(a) Show that T^* is linear and bounded and satisfied $\|T^*\| = \|T\|$.

(b) Suppose that T is onto Y and show that there exists $c > 0$ such that $\|T^*\phi\| \ge c\|\phi\|$ for all $\phi \in Y^*$.

Proof. (a) These are all Folland problems and have been proven elsewhere.

(b) It is also a Folland problem (Exercise 22c of Chapter 5) that whenever T is onto, T^* is injective. We can also show the range of T^* is closed in the following way. Let (ϕ_n) in the range of T^* converge to $\phi \in X^*$, and let $\psi_n \in Y^*$ be such that $T^*\psi_n = \psi_n \circ T = \phi_n$. Then $\psi_n \circ T \to \phi$ in X^*. Now

T is onto, so (ψ_n) is in fact Cauchy in Y^*, and therefore by completeness there exists $\psi = \lim_n \psi_n$. Since T^* is continuous by (a),

$$T^*\psi = \lim T^*\psi_n = \lim \phi_n = \phi.$$

This proves the claim. Hence the range of T^* is a Banach space in its own right, and the Banach isomorphism theorem applies and yields a bounded operator $(T^*)^{-1} : \text{ran}(T^*) \to Y^*$. Hence for $\phi \in Y^*$ there is some $\psi \in X^*$ such that $(T^*)^{-1}(\psi) = \phi$, and

$$\|(T^*)^{-1}(\psi)\| \leq \|(T^*)^{-1}\|\|\psi\| \Rightarrow \|T^*(\phi)\| \geq \frac{1}{\|(T^*)^{-1}\|}\|\phi\|.$$

□

Problem 7. *Let X be a compact Hausdorff space and let $C(X)$ be the Banach space of all continuous functions from X to $\mathbb{C}$, endowed with the usual uniform norm. Suppose $(f_n)_{n=1}^{\infty}$ is a bounded sequence in $C(X)$. Show that this sequence converges to 0 in the weak topology on $C(X)$ if and only if it converges pointwise to 0, namely,*

$$\forall x \in X \quad \lim_{n\to\infty} f_n(x) = 0.$$

Proof. The most frequently occurring qualifying exam problem has returned. :) See elsewhere, such as Problem 3 of August 2015. □

Problem 8. *Evaluate the limit*

$$\lim_{n\to\infty} \int_E \left(1 + \frac{x}{n}\right)^n \frac{e^{-x}}{(x^2-1)}\, d\lambda(x),$$

where λ is Lebesgue measure on $\mathbb{R}$ and $E = [2, \infty)$. Be sure to justify your assertions.

Using DCT. Note that

$$\left(1 + \frac{x}{n}\right)^n = e^{n\log\left(1+\frac{x}{n}\right)} \leq e^{\left(n\frac{x}{n}\right)} = e^x.$$

In the inequality above we use the fact that, for any $y > 0$, $\log(1 + y) < y$. (We see these two quantities are equal when $y = 0$, and $[\log(1 + y)] = \frac{1}{1+y} < 1 = y'$ when $y > 0$.)

Hence the sequence of functions $f_n(x) = \left(1 + \frac{x}{n}\right)^n \frac{e^{-x}}{x^2-1}$ is bounded above by $g(x) = \frac{1}{x^2-1}$. Using partial fraction decomposition (see bottom of MCT proof below) we see that $g \in L^1(E)$, so by DCT we have that

$$\frac{\partial}{\partial n}a_n(x) = (1 + \frac{x}{n})^n[\log(1 + \frac{x}{n}) - \frac{x}{n+x}] = (1 + \frac{x}{n})^n[\log(\frac{n+x}{n}) + \frac{n}{n+x} - 1] = \log(3)/2.$$

□

Using MCT. Claim: the sequence of functions

$$a_n(x) := \left(1 + \frac{x}{n}\right)^n$$

is increasing in n when $x \geq 2$. Take the partial derivatives with respect to n to get

$$\frac{\partial}{\partial n} a_n(x) = (1 + \frac{x}{n})^n [\log(1 + \frac{x}{n}) - \frac{x}{n+x}] = (1 + \frac{x}{n})^n [\log(\frac{n+x}{n}) + \frac{n}{n+x} - 1].$$

Now note the function $f(x) = \log(x) + \frac{1}{x} \geq 1$ for all $x \geq 1$. This is since $\log(1) + \frac{1}{1} = 1$ and $f'(x) = \frac{1}{x} - \frac{1}{x^2} = \frac{x-1}{x^2} > 0$ for $x > 1$. Hence

$$\log(\frac{n+x}{n}) + \frac{n}{n+x} - 1 \geq 0.$$

So $\frac{\partial}{\partial n} a_n(x) \geq 0$ for all $x \geq 2$, proving the claim. Now we may apply MCT and get

$$\lim_{n\to\infty} \int_E \left(1 + \frac{x}{n}\right)^n \frac{e^{-x}}{x^2 - 1}\, dx = \int_E \frac{1}{x^2-1}\, dx.$$

(We use that $\lim_{n\to\infty}(1 + \frac{x}{n})^n = \lim_{m\to\infty}(1 + \frac{1}{m})^{mx}$ by setting $m = \frac{n}{x}$.) We further note $\frac{1}{x^2-1} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$, so

$$\int_E \frac{1}{x^2-1}\, dx = \frac{1}{2} \int_E \left[\frac{1}{x-1} - \frac{1}{x+1}\right] dx = \frac{1}{2}[\log(x-1) - \log(x+1)]_2^\infty = \log(3)/2.$$

(We use that $\lim_{b\to\infty}[\log(b-1) - \log(b+1)] = 0$.) □

Problem 9. *(a) State the Principle of Uniform Boundedness.*

(b) Suppose that X and Y are real Banach spaces and that $\Phi : X \times Y \to \mathbb{R}$ is bilinear, meaning that it is linear in each variable separately. Suppose that

(i) for all $x \in X$ there exists $A_x \geq 0$ such that

$$\forall y \in Y \quad |\Phi(x, y)| \leq A_x \|y\|$$

(ii) for all $y \in Y$ there exists $B_y \geq 0$ such that

$$\forall x \in X \quad |\Phi(x, y)| \leq B_y \|x\|.$$

Show there is a constant $K \geq 0$ such that

$$\forall x \in X, \forall y \in Y \quad |\Phi(x, y)| \leq K \|x\| \|y\|.$$

Proof. (a) Folland. (b) Basically Folland, exercise 39 of Chapter 5. □

Problem 10. *For every natural number n, let $\mathcal{V}_n$ denote the subspace of all polynomials of degree $\leq n$, regarded as a subspace of $C[0, 1]$. Let*

$$\mathcal{V}_\infty = \bigcup_{n\geq 1} \mathcal{V}_n.$$

For which $n \in \{1, 2, \ldots, \infty\}$ does there exist a bounded linear functional ϕ on $C[0, 1]$ such that

$$\forall q \in \mathcal{V}_n \quad \phi(q) = q'(1)\,?$$

Justify your answer.

Proof. See Problem 4 of January 2013, noting the slight alteration. □

5 January 2022

Problem 1. *Prove or disprove that if $(f_n)_{n=1}^{\infty}$ is a sequence of Lebesgue integrable functions*

$$f_n : [0,1] \to \mathbb{R} \quad \textit{such that} \quad \lim_{n\to\infty} \|f_n\|_{L^1(\mathbb{R})} = 0,$$

then for at least one value $x \in [0,1]$ we have

$$\lim_{n\to\infty} f_n(x) = 0.$$

Proof. For $n = 2^m + k$ for $m \in \mathbb{N}_{\geq 0}$, $k \in [2^m]$, we define

$$f_n(x) = 1_{[\frac{k-1}{2^m}, \frac{k}{2^m}]}.$$

Then $f_n(x) \not\to 0$ for any x; indeed, $f_n(x) = 1$ for at least one value in $2^m \leq n < 2^{m+1}$. But $\int f_n = \frac{1}{2^m}$. This counterexample disproves the problem. □

Problem 2. *Let $\mathcal{A}$ be the set of all real-valued functions on $[0,1]$ for which $f(0) = 0$ and*

$$|f(t) - f(s)|^4 \leq t - s \quad \textit{for all} \quad 0 \leq s < t \leq 1.$$

Prove that $\mathcal{A}$ is a compact subset of $L^2[0,1]$. Don't forget to justify that $\mathcal{A}$ is closed in $L^2[0,1]$.

Proof. The set-up of the problem seems to suggest using Arzela-Ascoli I, but we are asking for this collection to be compact in $L^2[0,1]$, not $C[0,1]$. However, $\mathcal{A} \subset C[0,1]$! This is because if $s \to t$ in $[0,1]$, then $|f(t) - f(s)|^4 \to 0$ by the condition, so $|f(t) - f(s)| \to 0$ and $f(s)$ approaches $f(t)$. Also, convergence in $C[0,1]$ implies convergence in $L^2[0,1]$. (If $f_n \to f$ uniformly, it is easy to find a dominating function for f_n and use DCT: if we take $g := \max\{f_1, \ldots, f_{N+1}, |f_N| + 1\}$ where N is chosen such that $n > N$ implies $|f_n(x) - f(x)| < 1$ for all $x \in [0,1]$, then $|f_n| \leq g$ and g is in L^2.) So if we show $\mathcal{A}$ is compact in $C[0,1]$, then $\mathcal{A}$ is compact in $L^2[0,1]$ as well. (Continuous functions on a compact interval are bounded, so they are in L^2. If $(f_\alpha) \subset \mathcal{A}$, then there is a subnet converging to $f \in \mathcal{A} \subset C[0,1]$, so this subnet converges to f in $\mathcal{A} \subset L^2[0,1]$ as well by the above.) So Arzela-Ascoli I works fine; we will also need to show $\mathcal{A}$ is closed.

Let $\varepsilon > 0$ and choose $\delta < \varepsilon^4$. Then If $|t - s| < \delta$, for any $f \in \mathcal{A}$ we have $|f(t) - f(s)| \leq |t - s|^{1/4} < \varepsilon$. This guarantees equicontinuity, and in fact $|f(t)|^4 = |f(t) - f(0)|^4 \leq t \leq 1$ for all $t \in [0,1]$, so $|f(t)| \leq 1$, giving a pointwise (in fact uniform!) bound for functions in $\mathcal{A}$. Finally, if $f_n \to f$ uniformly for $(f_n) \subset \mathcal{A}$, we have $|f(t) - f(s)|^4 = \lim_n |f_n(t) - f_n(s)|^4 \leq t - s$ (since $|\cdot|$ and $(\cdot)^4$ are continuous functions), so $f \in \mathcal{A}$. □

Problem 3. *Let E be a subset of $\mathbb{R}$ that has positive Lebesgue measure and set*

$$S = \{f \in L^1(\mathbb{R}) : 1_E f = 0 \ \textit{a.e.}\}.$$

Prove or disprove that S is closed in $L^1(\mathbb{R})$.

Proof. Let $(f_n) \subset L^1(\mathbb{R})$ such that $1_E f_n = 0$ a.e. be chosen such that $f_n \to f \in L^1(\mathbb{R})$. We will attempt to show $f \in S$, thus proving the problem statement.

Let $(f_{n_k}) \subset (f_n)$ be chosen such that $f_{n_k} \to f$ a.e. Then since $f_{n_k} = 0$ a.e. on E and $f_{n_k} \to f$ a.e., in fact $f = 0$ a.e. on E as well. So $f \in S$. □

Problem 4. *Prove of disprove that there is a sequence of real polynomials $(p_n)_{n=1}^\infty$ such that*

$$\lim_{n\to\infty} \int_0^1 |p_n(t)|\,dt = 1,$$

but such that for all $t \in [0,1]$, $\lim_{n\to\infty} p_n(t) = 0$.

Proof. First note that

$$\int_0^1 x^n(1-x)\,dx = \frac{1}{n+1} - \frac{1}{n+2} = \frac{1}{(n+1)(n+2)}.$$

Define $f_n(x) := (n+1)(n+2)x^n(1-x)$; then by the above $\int_0^1 |p_n(x)|\,dx = 1$. Now x^n will dominate the behavior of this function for large enough n (polynomial growth is dominated by exponential growth) whenever $x < 1$, and $x^n \to 0$ in this case. Of course $f_n(1) = 0$ for all n, so we have proven the problem statement. □

Problem 5. *(a) Show (directly from the relevant definitions) that every separable metric space is a 2nd countable topological space.*

(b) Suppose μ is a Borel measure on a 2nd countable topological space X. Show that there exists a largest subset U of X that is both open and μ-null.

(c) Given an example of a topological space X and a Borel measure μ on X for which there is no largest subset of X that is both open and μ-null.

Proof. Before beginning we note the similarity between part (b) of this problem and Problem 2 of Chapter 7 in Folland. In particular, the complement of this largest subset is called the support of μ.

(a) Call this set X. Let $(x_i)_1^\infty$ be a countable dense subset of X, and consider the collection $(B(n^{-1}, x_i))_{i,n=1}^\infty$. This collection is certainly countable. For any $x \in X$, there is some sequence of (x_i) converging to x, so $x \in B(1, x_i)$ for large enough i. Also, if $x \in U$ for U open in X, then $B(\varepsilon, x) \subset U$ for some $\varepsilon > 0$. Choose x_i such that $d(x_i, x) < \varepsilon/2$ and pick n such that $n^{-1} < \varepsilon/2$; then $B(n^{-1}, x_i) \subset B(\varepsilon, x) \subset U$. So this is a countable basis for X.

(b) Let $\mathcal{C}$ be the collection of all open, μ-null sets in the countable basis for X. (If there are none, then by monotonicity $\mu > 0$ on all open sets, so $\varnothing$ is the desired largest subset.) Take the union of all elements C_i in $\mathcal{C}$ (which is countable!); then $\mu(\bigcup C_i) \le \sum_i \mu(C_i) = 0$. Suppose V is also open and μ-null; then $V = \bigcup U_j$ for U_j in the countable basis for X, and $\mu(U_j) \le \mu(V) = 0$ for all j. Hence $U_j \in \mathcal{C}$, so $\bigcup_i C_i \supset U_j$ for all j and therefore $\bigcup_i C_i \supset V$. This gives the largest subset we need.

(c) Let X be an uncountable set, and let $\mathcal{M}$ be the σ-algebra of countable and co-countable sets on X. Define $\mu : \mathcal{M} \to [0,1]$ to be 0 on countable sets and 1 on co-countable sets; then certainly $\mu(\varnothing) = 0$. What's more, for disjoint sets $(E_i)_1^\infty \subset \mathcal{M}$, *at most one* set E_i is co-countable as any such set is uncountable (since X is). Hence $\mu(\bigcup E_i)$ is 1 if one such sets E_i is co-countable and 0 otherwise (since the countable union of countable sets is countable). This agrees with the sum of the measures of these individual sets E_i, so μ is a measure.

Is there a largest μ-null set? If we suppose $U \in \mathcal{M}$ is the largest μ-null set, then U is countable by definition of μ, and since X is uncountable there is some other nonempty countable subset $V \subset U^c$, so $U \cup V$ is also μ-null, giving us our contradiction. □

Problem 6. *Let $(X, \mathcal{M}, \mu)$ be a finite measure space and suppose $f \in L^p(\mu)$ for some $p \in (0, \infty)$. Show*

$$\lim_{n\to\infty} \int |f|^{1/n}\, d\mu = \mu(\{x \in X : f(x) \neq 0\}).$$

Proof. See Problem 2 of January 2010, amongst others. □

Problem 7. *Suppose $(x_n)_{n=1}^\infty$ is a sequence in a Banach space X that converges weakly to $x \in X$. Show that*

$$\liminf_{n\to\infty} \|x_n\| \geq \|x\|.$$

Proof. Let f be the norm-one linear functional guaranteed by Hahn-Banach such that $f(x) = \|x\|$. Then $f(x_n) \to f(x) = \|x\|$ by definition of weak convergence. Note $|f(x_n)| \leq \|f\|\|x_n\| = \|x_n\|$. So if $(x_{n_k}) \subset (x_n)$ such that $\lim_k \|x_{n_k}\| = y$, then

$$y = \lim_k \|x_{n_k}\| \geq \lim_k |f(x_{n_k})| = \|x\|,$$

so any subsequential limit of $(\|x_n\|)$ is greater than or equal to $\|x\|$, completing the proof. □

Problem 8. *Let $1 \leq p \leq \infty$ and let $f \in L^p(\mathbb{R})$. Show that*

$$\int_{\mathbb{R}} \frac{|f(t)|}{1+t^2}\, dt < \infty.$$

Proof. By Hölder inequality, it suffices to show

$$\|\frac{1}{1+t^2}\|_q < \infty$$

where q is the conjugate exponent to p. Since $\{\frac{1}{1+t^2} : t \in \mathbb{R}\} \subset [0,1]$, $(\frac{1}{1+t^2})^q \leq (\frac{1}{1+t^2})$ for $q \in [1, \infty)$, so it further suffices to show

$$\int_{\mathbb{R}} \frac{1}{1+t^2}\, dt < \infty$$

(for the case where $q = \infty$, $\|\frac{1}{1+t^2}\|_\infty = 1 < \infty$ since $1 + t^2 \geq 1$). But this integral evaluates to $\arctan(t)|_{-\infty}^{\infty} = \pi$. □

Problem 9. *Show that for every positive integer n, there is a regular, Borel, signed measure μ on $[0,1]$ such that for all real polynomials P of degree $\leq n$,*

$$P'(1/2) = \int P(x)\, d\mu(x),$$

where P' is the derivative of P.

Proof. See Problem 4 of January 2013, noting the slight alteration. □

Problem 10. *Let X be the vector space of all real polynomials in one variable. Prove or disprove that there exists a norm on X making X into a Banach space.*

Proof. See Problem 9 of January 2010. □

6 August 2021

Problem 1. *Let (X, Ω) be a measurable space and suppose that $(f_n)_{n=1}^{\infty}$ is a sequence of real-valued measurable functions on X. Show that the set of all points $x \in X$ for which $(f_n(x))_{n=1}^{\infty}$ converges is a measurable set.*

Proof. Compare this with Exercise 3 in Chapter 2 of Folland.

We will say that a function may "converge to $\pm\infty$", although the result will similar even if we exclude this possibility. We have that $\limsup f_n(x)$, $\liminf f_n(x)$ are measurable functions. Define

$$g(x) = \begin{cases} 0 & \limsup f_n(x) = \infty = \liminf f_n(x) \text{ or } \limsup f_n(x) = -\infty = \liminf f_n(x) \\ \limsup f_n(x) - \liminf f_n(x) & \text{otherwise} \end{cases}.$$

Note that

$$\begin{aligned} g^{-1}(\{0\}) =& (\limsup f_n - \liminf f_n)^{-1}(\{0\}) \\ & \cup ((\limsup f_n)^{-1}(\{\infty\})) \cap (\liminf f_n)^{-1}(\{\infty\})) \\ & \cup ((\limsup f_n)^{-1}(\{-\infty\}) \cap (\liminf f_n)^{-1}(\{-\infty\})). \end{aligned}$$

The first set in this union is measurable since the subtraction of measurable functions is a measurable functions, and the other sets are measurable since the intersection of measurable sets is measurable. So $g^{-1}(\{0\}) = \{x : f_n \text{ converges}\}$ is measurable since the union of measurable sets is measurable. □

Problem 2. *Suppose that X is a linear subspace of $L^{2021}([0,1])$ that is closed as a subspace of $L^1([0,1])$. Show that X is closed as a subspace of $L^{2021}([0,1])$ and that $(X, \|\cdot\|_{2021})$ is isomorphic (meaning "linearly homomorphic") to a Hilbert space.*

Proof. We have that X is a subspace of $L^{2021}([0,1])$ with the following property: if $(f_n) \subset X$ and $f \in L^1$ such that $\|f_n - f\|_1 \to 0$, then $f \in X$. Consider $(f_n) \subset X$ and let $f \in L^{2021} \subset L^1$ (by a linear map) such that $\|f_n - f\|_{2021} \to 0$. Since $m([0,1]) = 1 < \infty$, then by Proposition 6.12 in Folland $\|f_n - f\|_1 \le \|f_n - f\|_{2021}$, so $\|f_n - f\|_1 \to 0$ as well, so $f \in X$ as desired. By a similar argument we have that if (f_n) is a sequence in X such that $\|f_n - f\|_2 \to 0$ for $f \in L^2 \subset L^1$, we get $f \in X$ as well. Hence X is closed as a subspace in L^2, so it may be linearly embedded as a Banach subspace of L^2. By restricting the inner product of L^2 to the space X, we find that X is isomorphic to a Hilbert space. □

Problem 3. *Regard $L^\infty(0,1) = L^1(0,1)^*$. Prove that if $f \in L^\infty(0,1)$, then there is a sequence $(p_n)_{n=1}^\infty$ of polynomials such that $(1_{(0,1)}p_n)_{n=1}^\infty$ converges weak* to f.*

Proof. Choose $f \in L^\infty$. By Lusin's theorem, there is a function $\phi_n \in C[0,1]$ and a set E such that $m(E) < 1/n$, $f = \phi_n$ on E^c, and $\|\phi_n\|_\infty \le \|f\|_\infty$. Let p_n be a polynomial such that $|\phi_n - p_n| < 1/n$. We note then that $|(p_n - f)(g)| \le (2(\|f\|_\infty) + 1)(|g|)$, which is an L^1 function. So

$$\lim_n \left| \int (p_n - f)(g) \right| \le \lim_n \left| \int_E (p_n - f)g \right| + \lim_n \int_{E^c} |(p_n - f)g| \le \int \lim_n |(p_n - f)g| 1_E + \lim_n \|g\|_1 \frac{1}{n} \to 0.$$

(The first term goes to zero as this integral is finite and the measure of E shrinks to zero.) The result follows. □

Problem 4. *Let a and b be real numbers satisfying $a > b > 1$. Evaluate*

$$\lim_{n\to\infty} \int_0^\infty \frac{n|\cos(x)|}{1 + n^a x^b}\, dx.$$

Proof. Let $y = nx$; then note that $dy = n\, dx$. So

$$\int_0^\infty \frac{n|\cos(x)|}{1 + n^a x^b}\, dx \le \int_0^\infty \frac{n}{1 + n^a x^b}\, dx = \int_0^\infty \frac{1}{1 + n^{a-b} y^b}\, dy.$$

There exists an $n > 0$ such that $n^{a-b} > 1$, in which case this function is bounded above by $\frac{1}{1+y^b}$. This function is integrable (by comparison and p-test on $[1,\infty)$ and by a bound above by 1 on $[0,1]$. So MCT applies (see the variant in Exercise 15 of Chapter 2 of Folland) and

$$\lim_{n\to\infty} \int_0^\infty \frac{n|\cos(x)|}{1 + n^a x^b}\, dx = \int_0^\infty \lim_{n\to\infty} \frac{|\cos(x)|n}{x^b n^a + 1}\, dx = \int_0^\infty 0\, dx = 0.$$

□

Problem 5. *(a) State the closed graph theorem.*

(b) Let $\alpha_n > 0$ (for $n \in \mathbb{N}$). Suppose that for any numbers $\gamma_n \ge 0$ we have

$$\sum_{n=1}^\infty \gamma_n^2 < \infty \quad \Rightarrow \quad \sum_{n=1}^\infty \frac{\gamma_n}{\sqrt{\alpha_n}} < \infty.$$

Show that we must have

$$\sum_{n=1}^\infty \frac{1}{\alpha_n} < \infty.$$

Proof. (a) See Theorem 5.12 of Folland. If X and Y are Banach spaces and $T : X \to Y$ is a closed linear map - i.e., if the graph of T is a closed subspace of $X \times Y$ - then T is bounded.

(b) Define $T_k : \ell^2 \to \mathbb{R}$ by

$$T_k((\gamma_n)_{n=1}^\infty) \to \sum_{n=1}^k \frac{\gamma_n}{\sqrt{\alpha_n}}.$$

This is clearly a linear map. It is also bounded by our assumption, which in fact tell us

$$\sup_k |T_k((\gamma_n)_{n=1}^\infty)| = \sum_{n=1}^\infty \frac{|\gamma_n|}{\sqrt{\alpha_n}} < \infty.$$

So by UBP

$$\begin{aligned}\infty > \sup\|T_k\| &\overset{\text{Riesz rep}}{=} \sup_k |\langle \cdot, \sum_{i=1}^k \frac{e_i}{\sqrt{\alpha_k}}\rangle| \\ &= \sup_k \|\sum_{i=1}^k \frac{e_i}{\sqrt{\alpha_k}}\|_2 = \sup(\sum_{i=1}^k \frac{1}{\alpha_k})^{1/2} \\ &= (\sum_{k=1}^\infty \frac{1}{\alpha_k})^{1/2}.\end{aligned}$$

So $\sum_{k=1}^\infty \frac{1}{\alpha_k} < \infty$ and we are done. (For the first equality on the second line, look at the paragraph above the Riesz representation theorem for Hilbert spaces in Section 5.5 of Folland.) □

Problem 6. *Prove that if X is a separable Banach space then there is an injective bounded linear operator from X into ℓ^{2021}.*

Proof. Let $(x_n)_{n=1}^\infty \subset X$ be dense in X. By Hahn-Banach there exists $x_n^* \in X^*$ such that $\|x_n^*\| = 1$ and $|x_n^*(x_n)| = \|x_n\|$. Define

$$T(x) = \sum_{k=1}^\infty \frac{x_k^*(x)}{2^k} e_k$$

for $T : X \to \ell^{2021}$. We want to show this operator is well-defined; it suffices to show it is bounded.

Let $x \in X$; then

$$\|T(x)\|_{2021} = (\sum_{k=1}^\infty \frac{|x_k^*(x)|^{2021}}{2^{2021k}})^{1/2021} \leq \|x\|(\sum_{k=1}^\infty \frac{1}{2^{2021k}})^{1/2021} \leq \|x\|.$$

(We are using $\|x_k^*\| = 1$.)

We now want to show T is injective. Let $(y_m)_{m=1}^\infty \subset \bigcup_{n=1}^\infty \{x_n\}$ be such that $y_m \to x$ where $0 = T(x) \Rightarrow x_k^*(x)$ for all $k \in \mathbb{N}$.

We claim $\|y_m\| \to 0$. We note there is some k_m such that $y_m = x_{k_m}$ for all m. So

$$\begin{aligned}0 = |x_{k_m}^*(x)| &\geq ||x_{k_m}^*(x - y_m)| - |x_{k_m}^*(y_m)| \\ &\Rightarrow |x_{k_m}^*(y_m)| = \|y_m\| \leq \|x_{k_m}^*\|\|x - y_m\| = \|x - y_m\| \to 0.\end{aligned}$$

Thus, $\|x\| \le \|x - y_m\| + \|y_m\| \to 0$. So $\|x\| = 0$ and we are done. □

Problem 7. *Prove that if $\mathcal{C}$ is a **weakly** compact subset of $C[0,1]$, then $\mathcal{C}$ is a **norm** compact subset of $L^2(0,1)$. You may use the theorem that if a subset of a Banach space is weakly compact then it is weakly sequentially compact.*

Proof. Take an arbitrary sequence $(f_n)_{n=1}^{\infty}$ in $\mathcal{C}$. By our assumption of weakly compactness of $\mathcal{C}$ there exists $f \in C[0,1]$ and a subsequence (f_{n_k}) such that $f_{n_k} \to f$ in $C[0,1]$. Let us replace f_{n_k} with f_n; we will show $f_n \to f$ in $L^2[0,1]$, completing the proof.

First we claim $\|f_n\|_\infty < \infty$. Define $T_n \in C[0,1]^{**}$ by $T_n(\mu) = \int f_n \, d\mu$. Note that $\forall \mu \in C[0,1]^*$,

$$|T_n(\mu)| \le |\int f_n - f \, d\mu| + |\int f \, d\mu| \le N + \|f\|_u \|\mu\|,$$

where N is some bound guaranteed by the fact that $\int f_n - f \, d\mu \to 0$; this sequence is eventually bounded, and we may exclude any elements of this sequence that are not. By UBP, $\sup_n \|T_n\| < \infty$. So there is some $M > 0$ such that $|T_n(\delta_x)| = |\int f_n \, d\delta_x| = |f_n(x)| \le M$. So $\|f_n\|_\infty \le M$, as we claimed.

Thus, as $|\int (f_n - f) \, d\delta_x| = |f_n(x) - f(x)| \to 0$ for all $x \in [0,1]$, and

$$|f_n - f|^2 \le (\|f_n\|_\infty + \|f\|_\infty)^2 \le (M + \|f\|_\infty)^2 \in L^1(0,1),$$

by LDCT, $\lim_{n\to\infty} \int |f_n - f|^2 \, dm = \int \lim_{n\to\infty} |f_n - f|^2 \, dm = 0$. □

Problem 8. *(a) Prove that every infinite-dimensional vector space contains a linearly independent set whose linear span is the whole space.*

(b) Prove that every infinite-dimensional Banach space has a discontinuous linear functional.

Proof. (a) Let $\mathcal{C}$ be the collection of linearly independent sets of our infinite-dimensional vector space. Any singleton set (other than 0) is linearly independent, so $\mathcal{C}$ is non-empty. Partially order $\mathcal{C}$ by inclusion and let (C_α) be a chain in $\mathcal{C}$. Consider $\bigcup_\alpha C_\alpha$ and assume $\sum_{i=1}^n \alpha_i c_i$ for $c_i \in \bigcup_\alpha C_\alpha$, α_i in our field. Then since (C_α) is a chain there is some C'_α containing all of c_i, so clearly all $\alpha_i = 0$. Hence by Zorn's lemma $\mathcal{C}$ has a maximal element C. Assuming the linear span of C is *not* the whole space, let v be an element in our vector space not in the linear span (note $v \neq 0$). Then we claim $C + v$ is linearly independent; indeed, if not, then there is some nontrivial solution (α_i) to $\sum_{i=1}^n \alpha_i \alpha_i c_i + \alpha_0 v$ for some $(c_i) \subset C$, $n \in \mathbb{N}$. But $\alpha_0 \neq 0$ since C is linearly independent, so we can write v as the linear span of the other elements of c_i - a contradiction. This proves the statement.

(b) Let $(x_n)_{n=1}^{\infty}$ be an infinite linearly independent set. WLOG we may assume $\|x_n\| = 1$ for all n. Define a linear function f on the subspace generated by these x_n such that $f(x_n) = n$. (We may extend linearly while keeping the function well-defined by definition of linear independence.) Then $\|f\| \ge n$ for all n, so f is unbounded and hence discontinuous. □

Problem 9. *Prove or disprove: There is a continuous function f from the reals to the reals such that for all rational numbers x, $f(x)$ is irrational, and for all irrational numbers x, $f(x)$ is rational.*

Proof. The following disproves the statement above. Assuming the existence of such a function f: for $q \in \mathbb{Q}$, the sets $f^{-1}(\{q\})$ are closed since f is continuous. Now by definition of f $\bigcup_{q\in\mathbb{Q}} f^{-1}(\{q\})$ is the set of irrational numbers. Hence $\mathbb{R} = \bigcup_{q\in\mathbb{Q}} f^{-1}(\{q\}) \cup \bigcup_{q\in\mathbb{Q}}\{q\}$, a countable union of nowhere dense sets. Baire Category Theorem gives the needed contradiction. □

Problem 10. *Let F be a continuous, real-valued function on $[0,1]\times[0,1]\times[-1,1]$ and for f in the unit ball of $C_{\mathbb{R}}[0,1]$, define $G_f : [0,1] \to \mathbb{R}$ by*

$$G_f(s) = \int_0^1 F(s,t,f(t))\,dt.$$

Show that $\{G_f : f \in C_{\mathbb{R}}[0,1], \|f\| \leq 1\}$ is a pre-compact subset of $C_{\mathbb{R}}[0,1]$.

Proof. It's time to use Arzela-Ascoli I. Note that F is continuous on a compact set, so it is uniformly continuous. Fix $\varepsilon > 0$ and choose $\delta > 0$ such that $d((s_1,t_1,u_1),(s_2,t_2,u_2)) < \delta$, then $|F(s_1,t_1,u_1) - F(s_2,t_2,u_2)| < \varepsilon$. Then for $|s_1 - s_2| < \delta$,

$$|G_f(s_1) - G_f(s_2)| = |\int_0^1 [F(s_1,t,f(t)) - F(s_2,t,f(t))]\,dt| \leq \int_0^1 |F(s_1,t,f(t)) - F(s_2,t,f(t))|\,dt < \varepsilon.$$

(In the first equality, by considering $F_1(t) := F(s_1,t,f(t))$ and $F_2(t) := F(s_2,t,f(t))$ we are free to write $\int F_1\,dt - \int F_2\,dt$ as $\int F_1 - F_2\,dt$.) This is true irrespective of our choice of f, so $\{G_f : f \in C_{\mathbb{R}}[0,1], \|f\| \leq 1\}$ is equicontinuous. Also $|F| \leq M$ for some $M > 0$, so $|G_f(s)| \leq \sup F(s,t,f(t))m([0,1]) \leq M$ as well, giving a pointwise bound. So Arzela-Ascoli applies and we are done. □

7 January 2021

Problem 1. *Let (X,μ) be a finite measure space and $f : X \to [0,\infty)$ an integrable function. For each n set $g_n(x) = f(x)^{1/n}$ for all $x \in X$. Show that the sequence (g_n) converges in $L^1(\mu)$ and determine the limit.*

Proof. See Problem 2 of January 2010, amongst others. □

Problem 2. *Let μ be Lebesgue measure on $[0,1]$ and let A be a closed subset of $[0,1]$. Prove that $\mu(A) = 0$ iff there is a sequence (p_n) of polynomials such that*

(i) $p_n(x) \geq 0$ *for all* n *and* $x \in [0,1]$,

(ii) $\int_0^1 p_n\,d\mu \to 0$ *as* $n \to \infty$, *and*

(iii) $p_n(x) \to \infty$ *for all* $x \in A$.

Proof. ($\Leftarrow$) Say we have polynomials $(p_n) \geq 0$ such that $\int p_n \to 0$ and $p_n(x) \to \infty$. Then $p_n \to 0$ in L^1, so there is some subsequence converging to 0 a.e. But $p_n(x) \to \infty$ for all x, so $\mu(A) = 0$.

($\Rightarrow$) Assume $\mu(A) = 0$ and A is closed. We will first show that (i), (ii), and (iii) hold where p_n is replaced with a continuous function f_n. Let $U_n \supset A$ be an open set such that $\mu(U_n \backslash A) < \frac{1}{n^2}$. Then

Urysohn guarantees a continuous $f_n : [0,1] \to [0,n]$ such that $f_n = n$ on A and $f_n = 0$ on U_n^c. Then

$$\int f_n = \int_A f_n + \int_{U_n \backslash A} f_n = n\mu(U_n \backslash A) \leq \frac{1}{n} \to 0.$$

This gives (ii), and (i) and (iii) are immediate by construction. For each n we may define a polynomial p_n such that $\|p_n - (f_n + \frac{1}{n})\|_\infty < \frac{1}{n}$; then

$$\int p_n \leq \int |p_n - (f_n + \frac{1}{n})| + \int (f_n + \frac{1}{n}) \leq \frac{1}{n}\mu[0,1] + \frac{1}{n} + \frac{1}{n}\mu[0,1] \to 0.$$

Since $p_n \geq f_n \geq n$ for all n, we are done. □

Problem 3. *(a) Let X be a normed space and (x_n) a sequence in X. For each n set $y_n = (x_1 + \cdots + x_n)/n$. Show that if (x_n) converges then so does (y_n).*

(b) Consider $[0,1]$ with Lebesgue measure μ. Show that there exists a sequence (f_n) of nonnegative integrable functions on $[0,1]$ such that f_n converges in measure to zero but the averages $g_n = (f_1 + \cdots + f_n)/n$ do not.

Proof. (a) Assume $x_n \to x$ in norm and define $y_n := (x_1 + \cdots + x_n)/n$. Then

$$\|y_n - x\| = \|\frac{x_1 - x}{n} + \cdots + \frac{x_n - x}{n}\| \leq \sum_{i=1}^{n} \frac{\|x_i - x\|}{n} = \sum_{i=1}^{N-1} \frac{\|x_i - x\|}{n} + \sum_{i=N}^{n} \frac{\|x_i - x\|}{n}$$

where $n > N$ and N is chosen such that $\|x_N - x\| < \frac{\varepsilon}{2}$ for all $n \geq N$. We can also choose $N_1 \geq N$ such that

$$\sum_{i=1}^{N-1} \frac{\|x_i - x\|}{N_1} < \frac{\varepsilon}{2}.$$

Then continuing from above, for $n \geq N_1$,

$$\|y_n - x\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(b) It's time for (an alteration of) your favorite sequence:

$$f_{2^n+i} = 2^n \mathbb{1}_{[\frac{i-1}{2^n}, \frac{i}{2^n}]} \text{ for } n \geq 0, 0 \leq i < 2^n.$$

We have $\{x : |f_{2^n+i}| > \varepsilon\} = \frac{1}{2^n}$, so this sequence goes to 0 in measure. However, we note that

$$\sum_{i=1}^{2^n-1} f_i = \sum_{k=0}^{k=n-1} 2^k \sum_{j=1}^{2^k} \mathbb{1}_{[\frac{j-1}{2^k}, \frac{j}{2^k}]} = \sum_{k=0}^{k=n-1} 2^k = 2^n - 1.$$

So there is a subsequence of f_{2^n+i} - namely (f_{2^n-1}) - where the averages are all equivalently the unit function, which does not converge in measure to zero. □

Problem 4. *Prove or disprove: ℓ_1 and c_0 are isomorphic. (Recall that the Banach space c_0 is the space of sequences $\mathbb{N} \to \mathbb{C}$ which conerge to zero, with pointwise vector space operations and supremum norm, and that an isomorphism between Banach spaces is an invertible bounded linear map with bounded inverse.)*

Proof. If ℓ_1 is isomorphic to c_0, then

$$\ell_\infty \cong \ell_1^* \cong c_0^* \cong \ell_1.$$

(1) $\ell_\infty \cong \ell_1^*$: dual of L_p results (section 6.2)

(2) $\ell_1^* \cong c_0^*$: if $T : \ell_1 \cong c_0$, we claim $T^\dagger : \ell_1^* \cong c_0^*$. We reference Exercise 22 of Chapter 5 of Folland to see $\|T\| = \|T^\dagger\|$ and $\|T^{-1}\| = \|(T^{-1})^\dagger\|$, then calculate

$$(T^{-1})^\dagger T^\dagger(g) = (T^{-1})^\dagger(g \circ T) = g,$$

and similarly for $T^\dagger(T^{-1})^\dagger$.

(3) By the Riesz representation theorem on $C_0(X)$, c_0^* is the set of complex Radon measures on $\mathbb{N}$, which have finite total variation (section 7.3).

But we claim $\ell_\infty \not\cong \ell_1$ because ℓ_∞ is not separable and ℓ_1 is. Indeed, consider ℓ_∞. For an arbitrary subset $K \subset \mathbb{N}$,

$$f_K(x) = \begin{cases} 1 & x \in K \\ 0 & x \notin K \end{cases}$$

is an uncountable family of elements each of distance 1 from each other, so there are uncountably many disjoint nonempty open balls in ℓ_∞. However, in the space ℓ_1, the rational span of the sequence

$$e_n = (0, \cdots, 0, \underbrace{1}_{n\text{th space}}, 0, \dots)$$

is countable and dense in ℓ_1. □

Problem 5. *Let (X, d) be a compact metric space and regard $C(X)^*$ as the space of finite Borel signed measures on X. Let (μ_n) be a weak* convergent sequence of Borel probability measures on X. Recall that the support of a measure on X is the complement of the union of all open sets with zero measure. Show that if the diameter of the support of μ_n tends to zero as $n \to \infty$ then the limit of (μ_n) is a point mass. Also, show that the converse is false.*

Proof. (1) We first note that $\mu(X) = 1$ since

$$\mu(X) = \int 1\, d\mu = \lim_n \int 1\, d\mu_n = \lim_n \mu_n(X) = 1.$$

(2) If $x \in \operatorname{supp}(\mu)$, then $\operatorname{dist}(x, \operatorname{supp}(\mu_n)) \to 0$. Suppose not. Define $A_n = \operatorname{supp}(\mu_n)$, $A = \operatorname{supp}(\mu)$. Then there is some $\varepsilon > 0$ such that for some subsequence (A_{n_k}), $\operatorname{dist}(x, A_{n_k}) \geq \varepsilon$ for all k. We will claim μ_n does *not* converge pointwise to μ given this assumption, which we will achieve by Urysohn's lemma.

Take $B = B(x, \varepsilon/2)$ and $B_1 = \overline{B(X, \varepsilon/4)}$. Note that $B_1 \subset B \subset A_{n_k}^c$ for all k. Then there is a function $f \in C(X, [0,1])$ which is 1 on B_1 and 0 on $B^c \supset A_{n_k}$. We have

$$\int f \geq \int_{B_1} f\, d\mu = \mu(B_1) > 0, \text{ but } \int d\mu_{n_k} = \int_{A_{n_k}} d\mu_{n_k} = 0.$$

(3) But there is only one such x that can satisfy $\text{dist}(x, \text{supp}(\mu_n)) \to 0$ since the diameter of $\text{supp}(\mu_n)$ goes to 0. Indeed, any two $x_1 \neq x_2$ are positive distance ε, and the diameter of $\text{supp}(\mu_n)$ is eventually less than $\varepsilon/3$, making this impossible.

(1)-(3) guarantee that μ is a point mass.

For the converse: define $\mu_n = (1 - \frac{1}{n})\delta_0 + \frac{1}{n}\delta_1$. Then since $f \in C[0, 1]$ is bounded,

$$\int f \, d\mu_n = (1 - \frac{1}{n})f(0) + \frac{1}{n}f(1) \to f(0) = \int f \, d\delta_0.$$

But the diameter of $\text{supp}(\mu_n) = \{0, 1\}$ is 1 for all n. □

Problem 6. *A sequence (x_n) in a normed space X is said to be weakly Cauchy provided that for each $x^* \in X^*$ the sequence $(x^*(x_n))$ is a convergent sequence of scalars.*

(a) Prove that a weakly Cauchy sequence in a normed space is norm bounded.

(b) Prove that a weakly Cauchy sequence in a reflexive Banach space is weakly convergent.

(a) This is a Uniform Boundedness problem. We have $\hat{x}_n(x^*) = x^*(x_n)$ is convergent for all $x^* \in X$. In particular, $\sup_n |\hat{x}_n(x^*)| < \infty$ for all $x^* \in X^*$, and since X^* is Banach $\sup_n \|\hat{x}_n\| < \infty$. The map $\hat{\cdot} : X \to X^{**}$ is an isometry, so we are done.

(b) Define $x^{**} \in X^{**}$ by $x^{**}(x^*) = \lim_n x^*(x_n)$ (for $x^* \in X^*$). We have x^{**} is bounded since

$$|x^{**}(x^*)| \leq \limsup_n \|x^*\| \|x_n\| \leq M\|x^*\|,$$

where $M = \sup\|x_n\| < \infty$ by (a). Since X is reflexive, there is some $x \in X$ such that $\hat{x} = x^{**}$. Then $x_n \to x$ weakly since

$$x^*(x_n) \to x^{**}(x^*) = \hat{x}(x^*) = x^*(x).$$

Problem 7. *Let K be a nonempty closed convex subset of $L_2(0, 1)$. Prove or disprove that there must exist an x in K such that $\|x\| = \inf_{y \in K} \|y\|$.*

Proof. Note that $L_2(0, 1)$ can be equipped with an inner product with which it becomes a Hilbert space. This turns out to be true, either by any solution to Exercise 59 in Folland or by Problem 5 on the January 2011 qualifying exam (which solves a more general problem). □

Problem 8. *Prove that if X is a separable Banach space then there is a bounded linear operator $T : \ell_2 \to X$ such that $T\ell_2$ is dense in X.*

Proof. Compare this to Exercise 36(b) in Chapter 5 of Folland; we are now tasked with constructing a similar map from L^2 and showing that such a map is *dense* in X rather than surjective. However, $\ell_1 \subset \ell_2$, so we need to be careful on how we form our function.

We define

$$T(f) = \sum_{n=1}^{\infty} \frac{f(n)}{2^n} x_n.$$

We show this series converges:

$$\sum_{n=1}^{\infty} \| \frac{f(n)}{2^n} x_n \| \le \sum_{n=1}^{\infty} \frac{|f(n)|}{2^n} \le \|f\|_2 \sum_{n=1}^{\infty} \frac{1}{2^n} = \|f\|_2 < \infty.$$

(Note $|f(n)|^2 \le \|f\|_2^2$.) T is clearly linear and bounded. To show $T\ell^2$ is dense in X, it suffices to show it contains (x_n). Define

$$e_n(k) = \begin{cases} 1 & n = k \\ 0 & n \ne k \end{cases}.$$

Then $T(2^n e_n) = x_n$. □

Problem 9. *Let (X, μ) be a finite measure space and let (A_n) be a sequence of measurable subset of X whose indicator functions χ_{A_n} converge in $L^1(\mu)$. Show that the limit is a.e. equal to the indicator function of some measurable set.*

Proof. Suppose $\chi_{A_n} \to f$ in L^1 for some $f \in L^1$. Then there is some subsequence $\chi_{A_{n_k}} \to f$ a.e. This (pointwise) limit must be 1 or 0, so f is an indicator function outside of a null set N. Note that $E := N^c \cap f^{-1}(\{1\})$ is measurable since Lebesgue measure is complete, and $f = \chi_E$ a.e. □

Problem 10. *Consider $[0, 1]$ with Lebesgue measure μ. For each n define*

$$f_n = \sum_{k=0}^{2^n-1} (-1)^k \chi_{A_k}$$

where $A_k = [k/2^n, (k + 1)/2^n]$. Show that $f_n \to 0$ weakly in $L^1[0, 1]$.

Slick proof with orthonormality. Compare this to the Rademacher functions in $\mathbb{R}$.

First note that that $L^2([0, 1]) \supset L^\infty([0, 1])$, so it suffices to show that $f_n \to 0$ in $L^2[0, 1]$.

However, the functions f_n are orthonormal functions in L^2! Indeed, if $n \ne m$, $f_n f_m$ is odd about $x = \frac{1}{2}$, and f_n^2 is the function 1. So we now only need to show that

$$\langle g, f_n \rangle \overset{n\to\infty}{\to} 0,$$

which is true because by Bessel's Inequality $\sum_{n=0}^{\infty} \langle g, f_n \rangle = \|g\|^2$, meaning the tail-end of this series goes to 0. □

More direct. The simple functions are dense in $L^\infty[0, 1]$ (see Theorem 2.10), so it suffices to show

$$\int_E f_n \to 0 \text{ for measurable sets } E \subset [0, 1].$$

Since the f_n are bounded and we can find an open set $U \supset E$ such that $\mu(U \backslash E) < \varepsilon$, it suffices to show

$$\int_a^b f_n \to 0 \, \forall a < b$$

(we need boundedness of $[0,1]$ as well). Let (a_j), (b_j) be sequences of dyadic rationals such that $a_j \searrow a$, $b_j \nearrow b$, and each of a_j, b_j have denominator 2^j. Choose i such that $a_j - a + b - b_j < \varepsilon/2$. Then since

$$|\int_a^b f_n| \leq \int_a^{a_j} |f_n| + \int_{b_j}^b |f_n| + \int_{a_j}^{b_j} |f_n| < |\int_{a_j}^{b_j} f_n| + \varepsilon,$$

it suffices to show $|\int_{a_j}^{b_j} f_n| \to 0$ (finally!).

If $n \geq j$, then $[a_j, b_j] = A_k \cup A_{k+1} \cup \cdots \cup A_{k+s}$ for some k, s, where the value of f on these A-sets alternates between ± 1. If $s - k + 1$ is even, the above integral is 0; otherwise it is in $\{-1/n, 1/n\}$. Either way,

$$|\int_{a_j}^{b_j} f_n| \leq \frac{1}{n} \to 0.$$

□

8 August 2020

Problem 1. *Let $f \in L^1(\mathbb{R})$. Stating any theorems that you use, compute*

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f(x)|^{1/n}\, dx.$$

Proof. See Problem 2 of January 2010, amongst others. □

Problem 2. *Let $f(x)$ be a real-valued continuous function on $[0,1]$ satisfying $f(0) = 0$. Given $\varepsilon > 0$, prove that there is a polynomial $p(x)$ such that*

$$\|f(x) - x^{1/2}p(x)\|_\infty < \varepsilon.$$

Proof. Our favorite theorem Stone-Weierstrass doesn't quite come into play until we find the right algebra to use it on. We decide to use the algebra:

$$A = \{g \in C[0,1] : g(x) = x^{1/2}h(x)\, \forall x \text{ for some } h \in C[0,1]\}.$$

Clearly this is a vector space, and $(x^{1/2}h_1(x))(x^{1/2}h_2(x)) = x^{1/2}(h_1(x)h_2(X)x^{1/2})$. Also $g(x) := x^{1/2}x^{1/2} = x \neq y = g(y)$ for $x \neq y$. We note $g(0) = 0$ for all g and apply Stone-Weierstrass to get

$$\bar{A} = \{g \in C[0,1] : g(0) = 0\}.$$

Now we can find a function $g = x^{1/2}h \in A$ such that $\|f - g\|_\infty < \frac{\varepsilon}{2}$ and a polynomial $p \in C[0,1]$ such that $\|p - h\|_\infty < \frac{\varepsilon}{2}$. So

$$\begin{aligned}\sup|f(x) - x^{1/2}p(x)| &\leq \|f-g\|_\infty + \sup_{x\in[0,1]} |x^{1/2}h(x) - x^{1/2}p(x)| \\ &\leq \|f-g\|_\infty + \sup_{x\in[0,1]} |x^{1/2}| \cdot \sup_{x\in[0,1]} |h(x)-p(x)| \\ &= \|f-g\|_\infty + 1\|h-p\|_\infty < \varepsilon.\end{aligned}$$

□

Problem 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function such that*

$$\int_a^b f(x)\,dx = 0 \text{ for every } a < b.$$

Show that $f(x) = 0$ for almost every $x \in \mathbb{R}$.

Proof. It doesn't seem non-trivial to invoke Lebesgue Differential Theorem here: i.e., since $f \in L^1 \subset L^1_{\text{loc}}$ we have

$$0 = \lim_{n\to\infty} \frac{n}{2} \int_{x-1/n}^{x+1/n} f(y)\,dy = f(x)$$

for a.e. x. However, we might just want to feel better that we maybe didn't think of this and just say we thought this other solution was "better suited" for a qual problem :)

Alternative solution: We want to show $\mu(\{x : f(x) \neq 0\}) = 0$. Let us assume $f \geq 0$, set $E := \{x : f(x) > 0\}$ and further assume $\mu(E) > 0$. We can certainly find a compact set $F \subset [-m, m]$ contained in E such that $\mu(F) > 0$. We note $U := F^c \cap (-m, m)$ is open and hence can be written as a union of disjoint intervals $\bigcup_n (a_n, b_n)$ Then

$$0 = \int_{-m}^{m} f(x)\,dx = \int_F f + \int_U f = \int_F f + \sum_{n=1}^{\infty} \int_{a_n}^{b_n} f = \int_F f.$$

This contradicts our choice of F. For general f we may apply this on f^+ and f^-, as both are also L^1. □

Problem 4. *Let f be Lebesgue integrable on $(0, 1)$. For $0 < x < 1$ define*

$$g(x) = \int_x^1 t^{-1} f(t)\,dt.$$

Prove that g is Lebesgue integrable on $(0, 1)$ and that

$$\int_0^1 g(x)\,dx = \int_0^1 f(x)\,dx.$$

(Hint: first prove the claim under the assumption that $f(x) \geq 0$.)

Proof. If $f \geq 0$, one can use Tonelli below:

$$\int_0^1 g(x)\,dx = \int_0^1 \left(\int_x^1 t^{-1} f(t)\,dt \right) dx$$
$$= \int_0^1 \left(\int_0^t t^{-1} f(t)\,dx \right) dt$$
$$= f(t)\,dt < \infty.$$

Now that we know g is L_1 for any integrable f, Fubini also allows us to make the above calculation for the general case. ☐

Problem 5. *Let X be an infinite-dimensional Banach space. Show that the weak closure of the sphere $S_X = \{x \in X : \|x\| = 1\}$ is the unit ball $B_X = \{x \in X : \|x\| = 1\}$.*

Proof. This is very similar to Exercise 63 in Chapter 5 of Folland. We will need to make a few adjustments to the standard proof for this exercise.

First, we know from Exercise 48 of Chapter 5 of Folland (or Mazur's theorem, if you are familiar) that the norm-closed ball B_X is also weakly closed, so $\overline{S_X}^w \subset B_X$. (You may want to prove it.)

To show $B_X \subset \overline{S_X}^w$, fix $x \in B_X$ and let V be a weak neighborhood of x. Then

$$V \supset (x_1^*)^{-1}(V_1) \cap \cdots \cap (x_n^*)^{-1}(V_n)$$

for some $x_i^* \in X^*$, $V_i \subset \Bbbk$ open s.t. $x_i^*(x) \in V_i$ for all $i \in [n]$.

Since X is infinite-dimensional, there is some $y \neq 0$ such that $y \in \ker(x_i^*)$ because $\bigcap_{i=1}^n \ker(x_i^*)$ has finite codimension. Then for all $t \in \Bbbk$, $x + ty \in V$ since $x_i^*(x+ty) = x_i^*(x) \in V_i$.

The mapping $t \mapsto \|x + ty\|$ is continuous, and when $\|x + ty\| = \|x\| \leq 1$. As $t \to \infty$, $\|x + ty\| \to \infty$, so by intermediate value theorem there is some $t_0 \in \Bbbk$ such that $\|x + t_0 y\| = 1$. Therefore $V \cap S_X \supset \{x + t_0 y\} \neq \varnothing$. Hence $x \in \overline{S_X}^w$, yielding the desired equality. ☐

Problem 6. *Let (A_k) be a sequence of measurable subsets of a measure space $(X, \mathcal{M}, \mu)$ and let B_m be the set of all $x \in X$ which are contained in at least m of the sets A_k, $k \in \mathbb{N}$.*

Prove that B_m is measurable and that

$$\mu(B_m) \leq \frac{1}{m} \sum_{k=1}^{\infty} \mu(A_k).$$

Proof. See Problem 3 of August 2019. ☐

Problem 7. *(a) State Tietze's Extension Theorem.*

(b) Let $n \in \mathbb{N}$ and let $(x_j)_{j=1}^n \subset [0,1]$ and $(r_j)_{j=1}^n \subset \mathbb{R}$ be given. Show that there is a continuous function $f : [0,1] \to \mathbb{R}$ with the property that $f(x_j) = r_j$, $j \in [n]$, and

$$\int_0^1 f(x)\,dx = 0.$$

Proof. (a) See Theorem 4.16 in Folland. If X is a normal topological space, $A \subset X$ is closed, and $f : A \to [a, b]$ is continuous, there exists a continuous extension $\tilde{f} : X \to [a, b]$. (We note that there is an LCH version of this result as well.)

(b) Tietze is not necessary here; although there would exist a continuous function such that $f(x_j) = r_j$ by Tietze, more work would need to follow to get such a function where $\int f = 0$, so we take a different approach. Choose $\varepsilon > 0$ such that $[x_i - 3\varepsilon, x_i + 3\varepsilon] \cap [x_j - 3\varepsilon, x_j + 3\varepsilon] = \varnothing$ for all $i \neq j$. We can further choose ε such that for all $x_i \neq 0, 1$, $[x_i - 3\varepsilon, x_i + \varepsilon] \subset [0, 1]$.

Let $f \equiv 0$ on $(\bigcup_i [x_i - 3\varepsilon, x_i + 3\varepsilon])^c$. If $x_i \neq 0, 1$, define

$$f(x)|_{[x_i - 3\varepsilon, x_i + 3\varepsilon]} = \begin{cases} 0 & x_i - 3\varepsilon \leq x \leq x_i - \varepsilon \\ -\frac{r_i}{\varepsilon}(x_i - \varepsilon) + \frac{r_i}{\varepsilon}(x) & x_i - \varepsilon \leq x \leq x_i \\ \frac{r_i}{\varepsilon}(x_i + \varepsilon) - \frac{r_i}{\varepsilon}(x) & x_i \leq x \leq x_i + 2\varepsilon \\ \frac{r_i}{\varepsilon}(x_i + 3\varepsilon) + \frac{r_i}{\varepsilon}(x) & x_i + 2\varepsilon \leq x \leq x_i + 3\varepsilon \end{cases}.$$

If $x_1 = 0$, define

$$f(x)|_{[0,3\varepsilon]} = \begin{cases} r_1 - \frac{r_1}{\varepsilon}x & 0 \leq x \leq \varepsilon \\ \frac{r_1}{2} - \frac{r_1}{2\varepsilon}x & \varepsilon \leq x \leq 2\varepsilon \\ -\frac{3r_1}{2} + \frac{r_1}{2\varepsilon}x & 2\varepsilon \leq x 3\varepsilon \end{cases}.$$

Similarly if $x_n = 1$. (This is a zero function with $2n$ spikes such that the integral of each spike is the negative of its adjacent spike.) Then f is continuous, and by construction $\int_0^1 f = 0$. ☐

Problem 8. *(a) Show that $C[0, 1]$ can be naturally viewed as a subspace of $L^2[0, 1]$ (on $[0, 1]$ we consider the Lebesgeu [it's a typo but I'm not changing it] measure) by proving that each equivalence class in $L^2[0, 1]$ contains at most one function in $C[0, 1]$ (fixed an actual exam typo).*

(b) Let $T : L^2(\mu) \to L^2(\mu)$ be a bounded linear map satisfying $T(C([0, 1])) \subset C([0, 1])$. Show that the map $f \mapsto T(f)$ from $C[0, 1]$ to itself is bounded with respect to the supremum norm.

Proof. (a) We only need to show that, whenever $f, g \in C[0, 1]$, ($f = g$ a.e.)$\Rightarrow$($f = g$). We know from theorems in Folland that $\int |f - g| = 0$, meaning that $\{x : |f - g| > 0\}$ has zero measure. But this is an open set since $|f - g|$ is continuous, so it is empty.

(b) We will use Closed Graph Theorem and show the graph of $T|_{C[0,1]}$ is closed in $C[0, 1] \times C[0, 1]$. Suppose (f_n, Tf_n) is a sequence in the graph of this function such that

$$(f_n, Tf_n) \to (f, g) \text{ in } C[0, 1] \times C[0, 1].$$

We want to show $g = Tf$. Since T is bounded in the L^2-norm, we can calculate

$$\|g - Tf\|_2 \leq \|g - Tf_n\|_2 + \|Tf_n - Tf\|_2.$$

This first term goes to 0 since $f_n \to g$ in supremum norm and hence similarly in L^2-norm. The second term also goes to zero since T is continuous on L^2. So $g = Tf$ a.e. and by (a) $Tf = g$. So $T|_{C[0,1]}$ is $C[0, 1]$-continuous. ☐

Problem 9. *(a) Let $C[0,1]$ be the Banach space of real-valued continuous functions on $[0,1]$. Find the extreme points of the unit ball of $C[0,1]$.*

(b) Show that $C[0,1]$ is not isometrically isomorphic to a dual space of a Banach space.

Proof. See Problem 10 of August 2010. □

Problem 10. *Let μ be a Borel measure on $[0,1]$ with $\mu([0,1]) = 1$.*

(a) Show that if μ is atomless, then for any $0 < r < 1$ there is a measurable $A \subset [0,1]$ with $\mu(A) = r$.

Recall that $A \subset [0,1]$ is an <u>atom</u> for μ if $\mu(A) > 0$, and for all measurable $B \subset A$, either $\mu(B) = \mu(A)$ or $\mu(B) = 0$.

(b) Show that μ is atomless iff for each $n \in \mathbb{N}$ there is a partition of $[0,1]$ into n sets $A_1, \dots, A_n$ with $\mu(A_j) = \frac{1}{n}$ for $j \in [n]$.

Proof. (a) Define a function $f : [0,1] \to [0,1]$ using the rule $f(x) = \mu([0,x])$. The function f is well-defined since μ is Borel, f is increasing, $f(0) = 0$, and $f(1) = 1$. If we show that f is continuous, we are done by Intermediate Value Theorem.

Since f is increasing, the only discontinuity f can have is a jump discontinuity. Say there exists $c \in (0,1]$ such that $\lim_{x\to c^-} f(x) \neq f(c)$. Then for sufficiently large n, $f(c) - f(c - \frac{1}{n}) = \mu((c - \frac{1}{n}, c])$ is well-defined, and $\lim_{n\to\infty} f(c) - f(c - \frac{1}{n}) > 0$. Hence $\mu(\{c\}) = \lim_{n\to\infty} \mu((c - \frac{1}{n}, c]) > 0$. But then $\{c\}$ is an atom, contradicting the fact that μ is atomless. A similar argument shows that $\lim_{x\to c^+} f(x) = f(c)$ for all $c \in [0,1)$, showing that f is continuous.

(b) ($\Rightarrow$) We can find a set $A_1 \subset [0,1]$ of $\mu(A_1) = \frac{1}{n}$. For $k < n-1$, having found A_k we can find $A_{k+1} \subset [0,1] \backslash (\bigcup_i A_i)^c$ such that $\mu(A_{k+1}) = \frac{1}{n}$. Then setting $A_n = (\bigcup_i A_i)^c$ gives us our result.

($\Leftarrow$) Let $\mu(A) > 0$ and pick n such that $\frac{1}{n} < \mu(A)$. Then $\mu(A \cap A_j) \neq 0$ for some j. Since $\mu(A \cap A_j) \leq \frac{1}{n} < \mu(A)$, we are done. □

9 January 2020

Problem 1. *Show that*

$$\lim_{n\to\infty} \int_0^\infty \frac{4t^3 + 12}{12t^6 + 3nt + 2}\, dt = 0.$$

Proof. First, it is clear that

$$f_n(t) := \frac{4t^3 + 12}{12t^6 + 3nt + 2} \to 0$$

pointwise. We also have

$$f_n(t) \leq \frac{4t^3 + 12}{12t^6} \in L^1(1, \infty)$$
$$f_n(t) \leq \frac{4t^3 + 12}{12t^6 + 2} \in L^1(0, 1).$$

(The first equation is dominated by $\frac{4}{3t^3}$ on $(1, \infty)$; the other by 8 on $(0, 1)$.)

In light of this we define

$$g(t) = \begin{cases} \frac{4t^3+12}{12t^6+2} & t \in (0, 1] \\ \frac{4t^3+12}{12t^6} & t \in (1, \infty) \end{cases}.$$

Then $|f_n| \leq g \in L^1(0, \infty)$. DCT completes the proof. □

Problem 2. *Show for all $f \in L_1(\mathbb{R})$ that*

$$\lim_{\delta \to 1} \int |f(\delta x) - f(x)|\, dx = 0.$$

Proof. We claim $\|f(\delta x)\|_1 \leq \frac{1}{\delta}\|f\|_1$. Note this is true for indicator functions on sets $[a, b]$ by u-substitution

$$\int 1_{[a,b]}(\delta x)\, dx = \frac{1}{\delta} \int 1_{[a,b]}(x)\, dx = \frac{1}{\delta} m([a, b]),$$

so it is true for simple functions and hence for all L_1 functions. In particular, this means that $|f(\delta x) - f(x)| \leq |f(\delta x)| + |f(x)| \in L_1(\mathbb{R})$. If $f \in L_1$ is continuous, then DCT implies

$$\lim_{\delta \to 1} \int |f(\delta x) - f(x)|\, dx = \int |f(\lim_{\delta \to 1} \delta x) - f(x)|\, dx = 0,$$

and since the continuous functions are dense in L_1 we are done.

(One can also do this by assuming f is of compact support to get a similar L_1-bound for DCT.) □

Problem 3. *For an integrable function $f \in L_1(\mathbb{R})$, and $\alpha \geq 0$ put*

$$E_\alpha := \{x \in \mathbb{R} : |f(x)| \geq \alpha\}.$$

Show that the map $\alpha \mapsto m(E_\alpha)$ is measurable and that

$$\int_{-\infty}^{\infty} |f(x)|\, dx = \int_0^\infty m(E_\alpha)\, d\alpha.$$

Proof. Fix $f \in L_1(\mathbb{R})$. Define $g : [0, \infty) \to \mathbb{R}$ by

$$g(\alpha) = m(E_\alpha).$$

We want to show that g is measurable; i.e., we want

$$g^{-1}(-\infty, b) : \{\alpha \in [0, \infty) : g(\alpha) < b\}$$

to be measurable for all $b \in \mathbb{R}$. First: $E_0 = \mathbb{R}$, and if $\alpha < \alpha'$, then $E_{\alpha'} \subset E_\alpha$. Hence if $c \in g^{-1}(-\infty, b)$, then if $c' > c$ we have

$$g(c') \leq g(c) < b.$$

Therefore $g^{-1}(-\infty, b)$ is an interval, so it is Lebesgue measurable. Hence g is measurable.

To finish the problem, note that

$$\int_0^\infty m(E_a)\,da = \int_0^\infty \int_{\mathcal{R}} 1_{\{x:|f(x)|\geq a\}}\,dx\,da \overset{\text{Tonelli}}{=} \int_{-\infty}^\infty \int_0^\infty 1_{\{x:|f(x)|\geq a\}}\,da\,dx = \int_{-\infty}^\infty |f(x)|\,dx.$$

□

Problem 4. *Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Using the inequality $a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda)b$, for $0 < \lambda < 1$ and $0 \leq a, b$, prove the Hölder inequality.*

Proof. This proof is given as the proof of Theorem 6.2 in Folland. We restate the Hölder inequality (the author does not anticipate any equality stipulations require stating or proof): if f, g are measurable functions on X,

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

If $\|f\|_p$ or $\|g\|_q = 0$ - or if $f \notin L_p$ or $g \notin L_q$ - this is obvious. If $\|f\|_p = 1 = \|g\|_q$, then

$$\begin{aligned}\|fg\|_1 &= \int_X |f(x)||g(x)|\,d\mu = \int_X (|f(x)|^p)^{1/p}(|g(x)|^q)^{1/q}\,d\mu \\ &\overset{a^\lambda b^{1-\lambda}}{\leq} \int_X \frac{|f(x)|^p}{p}\,d\mu + \int \frac{|g(x)|^q}{q}\,d\mu \\ &= \frac{1}{p}\|f\|_p^p + \frac{1}{q}\|g\|_q^q = 1.\end{aligned}$$

In general,

$$\|fg\|_1 = \|f\|_p\|g\|_q \|\frac{f}{\|f\|_1}\frac{g}{\|g\|_q}\| \leq \|f\|_p\|g\|_q.$$

□

Problem 5. *Show that for $\varepsilon > 0$ there is a closed subset $E \subset [0,1]$ with empty interior, of Lebesgue measure at least $1 - \varepsilon$.*

Proof. The generalized Cantor set is given as an answer for this problem in Folland's text, and it is up to us to show such a set has the intended measure (see Exercise 32 in Chapter 1 of Folland).

We reiterate: take a sequence of real numbers (α_i) such that $\sum \alpha_i < \infty$. The construction of this set comes from removing the (open) middle α_ith of each interval present at the $i-1$th step of our construction (instead of the middle third) to get a set E_i. The intersection $\bigcap_{i=1}^\infty E_i$ will have measure $\prod_j (1-\alpha_j)$, since removing α_ith of the set leaves us with $(1-\alpha_i)$th of it (apply an induction argument). Clearly the remaining set will be closed, as it is an intersection of closed sets. Since the length of the longest (and every) interval is at least halved at each step, it will also have empty interior. We will show there exists a sequence (α_i) such that $\prod_i (1-\alpha_i) = 1 - \frac{\alpha}{1-2\alpha}$ for $\alpha \in (0, \frac{1}{3})$, completing the proof.

Define $\alpha_i := \alpha^i$. Then an easy induction argument shows that

$$\begin{aligned}m(E_n) &= 1 - \sum_{k=1}^n 2^{k-1}\alpha^k = 1 - \frac{1}{2}\sum_{k=1}^n (2\alpha)^k \\ &\overset{n\to\infty}{\to} 1 - \frac{1}{2}(\frac{2\alpha}{1-2\alpha}) = 1 - \frac{\alpha}{1-2\alpha}.\end{aligned}$$

□

Problem 6. *Let X be a Banach space and Y a non-trivial closed subspace of X.*

(a) Show that for all $y^* \in Y^*$ (the dual of Y) the set

$$\{x^* \in X^* : \|x^*\| = \|y^*\| \text{ and } x^*|_Y = y^*\}$$

is weak-compact.*

(b) Show that every extreme point of the closed unit ball of Y^* extends to an extreme point of the unit ball of X^*.

Proof. (a) Define

$$E_{y^*} := \{x^* \in X^* : \|x^*\| = \|y^*\| \text{ and } x^*|_Y = y^*\}.$$

Note $E_{y^*} \subset \|y^*\| B_{X^*}$, so by Alaoglu it suffices to show that E_{y^*} is weak*-closed.

To this end, suppose that (x^*_α) is a net in E_{y^*} converging weak* to x^*. If $y \in Y$, then noting that weak*-convergence is pointwise convergence:

$$x^*(y) = \lim x^*_\alpha(y) = \lim_\alpha y^*(y) = y^*(y).$$

Hence $x^*|_Y = y^*$. Y is non trivial, so $\|x^*\| \geq \|y^*\|$. But since $(x^*_\alpha) \subset E_{y^*} \subset \|y^*\| B_{X^*}$, which is weak*-closed, we have $\|x^*_\alpha\| \leq \|y^*\|$. Since the norm is calculated as the supremum of point-norms, we get $\|x^*\| \leq \|y^*\|$, so E_{y^*} is weak*-closed and hence weak*-compact.

(b) Suppose y^* is an extreme point of B_{Y^*}. Then $\|y^*\| = 1$ and E_{y^*} is weak*-compact. By Hahn-Banach $E_{y^*} \neq \varnothing$. E_{y^*} is convex because if $x^*_1, x^*_2 \in E_{y^*}$ and $\lambda \in [0,1]$, then

$$(\lambda x^*_1 + (1-\lambda)x^*_2)|_Y = \lambda x^*_1|_Y + (1-\lambda)x^*_2|_Y = \lambda y^* + (1-\lambda)y^* = y^*.$$

Since Y is nontrivial, $\|\lambda x^*_1 + (1-\lambda)x^*_2\| \geq \|y^*\|$, and also

$$\|\lambda x^*_1 + (1-\lambda)x^*_2\| \leq \lambda\|x^*_1\| + (1-\lambda)\|x^*_2\| = \|y^*\|.$$

So $\lambda x^*_1 + (1-\lambda)x^*_2 \in E_{y^*}$. We have gathered that E_{y^*} is convex, nonempty, and weak*-compact. So by Krein-Milman, E^*_y is the weak*-closure of the convex hull of its extreme points.

Now for any extreme point of $y^* \in B_{y^*}$, we know E_{y^*} is non-empty, so take an extreme point x^* on E_{y^*}. We claim x^* is an extreme point of B_{X^*}, and since x^* extends y^* and maintains its norm we are done. Let $x^* = \lambda x^*_1 = (1-\lambda)x^*_2$ for $\lambda \in (0,1)$ and $x^*_1, x^*_2 \in B_{X^*}$. Then $x^*_1|_Y, x^*_2|_Y \in B_{Y^*}$ and $y^* = x^*|_Y = \lambda x^*_1|_Y + (1-\lambda)x^*_2|_Y$, so $x^*_1|_Y = x^*_2|Y = y^*$. Clearly $\|x^*_1\| = \|x^*_2\| = 1 = \|y^*\|$, so in fact $x^*_1, x^*_2 \in E_{y^*}$. But x^* is an extreme point of E_{y^*}, so x^* is an extreme point of B_{X^*}. □

Problem 7. *(I refuse the notation from the exam and have substituted my own.) Assume that $(X, \|\cdot\|_1)$ is a normed linear space and that Y is a subspace of X. Assume that $\|\cdot\|_2$ is a norm on Y which is equivalent to $\|\cdot\|_1$. Prove that $\|\cdot\|_2$ can be extended to an equivalent norm on all of X.*

Proof. The following proof uses some results from convex analysis. One may review Section 4.1 of Conway's *A Course in Functional Analysis* as well as Chapter 1 of Rudin's *Functional Analysis*.

Denote by $B = B_{(X,\|\cdot\|_1)}$ and $B' = B_{(Y,\|\cdot\|_2)}$. There exist $0 < c < c'$ such that

$$c\|y\|_1 \leq \|y\|_2 \leq c'\|y\|_1.$$

Thus,

$$cB \cap Y \subset B' \subset c'B \cap Y.$$

Let $K := \text{conv}(B' \cup cB)$. K is convex and contains B', so since X is locally convex it is a neighborhood of 0. It is also balanced (i.e., $x \in K \Rightarrow \alpha x \in K$ for $|\alpha| \leq 1$) because B' and cB are (in particular, linear scaling is easy since convex, and scalar rotation is possible since multiplying $k = tb_1 + (1-t)b_2$ for $b_i \in B' \cup cB$ by $|\alpha| = 1$ gives $\alpha k = t(\alpha b_1) + (1-t)(\alpha b_2)$ - proceed by induction on number of terms). Also, $c'B \subset B' \cup cB$, and since $c'B$ is convex we have

$$cB \subset K \subset c'B,$$

so K is bounded. We now invoke the following theorem:

Let $K \subset V$ be a convex, bounded, balanced neighborhood of 0. Then

$$\mu_K := \inf\{r > 0 : x \in rK\}$$

is a norm on V, and $\|\cdot\| = \mu_{B_V}$.

(We know the seminorm μ_K is a norm here since V is a normed space. Here V may be X or Y; we will use it in both cases.) We now split the rest of the proof into two:

(1) μ_k is equivalent to $\|\cdot\|$. Observe that

$$c\|\cdot\|_1 = c\mu_B = \mu_{cB} \leq \mu_K \leq \mu_{c'B} = c'\mu_B = c'\|\cdot\|_1.$$

(2) μ_k extends $\|\cdot\|_2$ on Y. We want to verify that $K \cap Y = B'$. Clearly $B' \subset K \cap Y$; to show the opposite inclusion, suppose $y \in K \cap Y$. Then

$$y = \lambda y_1 + (1-\lambda)y_2$$

for some $\lambda \in [0,1]$, $y_1 \in B'$, $y_2 \in cB$. Easy to see if $\lambda = 1$; otherwise

$$y_2 = \frac{1}{1-\lambda}(y - \lambda y_1) \in Y \cap cB \subset B'.$$

So $y \in B'$ since B' is convex, and $K \cap Y = B'$. Therefore

$$\mu_k(y) = \inf\{r > 0 : y \in rK\} = \inf\{r > 0 : y \in rB'\} = \mu_{B'}(y) = \|y\|_2.$$

□

Problem 8. *Let (f_n) be a sequence of continuous functions on $[0,1]$, such that for each $x \in [0,1]$ there is an $n_x \in \mathbb{N}$, so that $f_n(x) \geq 0$ for all $n \geq n_x$.*

Show that there are an $N \in \mathbb{N}$ and an open nonempty interval $I \subseteq [0,1]$, so that $f_n(x) \geq 0$ for all $n \geq N$ and $x \in I$.

Proof. Baire Category Theorem time! Define

$$F_n := \{x : f_n(x) \geq 0\} = f_n^{-1}[0, \infty) \quad \text{and} \quad E_n = \bigcap_{k=n}^{\infty} F_n.$$

Then (E_n) is a sequence of closed sets, and what's more, since for any x there is some n_x such that $f_n(x) \geq 0$ for all $n \geq n_x$,

$$[0, 1] = \bigcup_{n=1}^{\infty} E_n.$$

Since $[0, 1]$ is not a countable union of nowhere dense sets, there is some N such that E_N has nonempty interior. Pick an interval $I \subset E_N$. Then since the (E_n) are nested, for all $x \in I$ $f_n(x) \geq 0$ for all $n \geq N$. □

Problem 9. *For a bounded sequence $(f_n) \subset C[0, 1]$, show that*

$$f_n \to_{n\to\infty} 0 \text{ weakly} \iff f_n(x) \to_{n\to\infty} 0 \text{ for all } x \in [0, 1].$$

Proof. See Problem 4 of August 2010. □

Problem 10. *On the set $[0, \infty]$ consider the topology $\mathcal{T}$ generated by the open sets (in the usual topology) of $[0, \infty)$ and the sets of the form $[0, \infty]\backslash C$, with $C \subset [0, \infty)$ compact.*

(a) Show that $[0, \infty]$ with above defined topology is a compact space.

(b) Show that $[0, \infty]$ with above defined topology is metrizable. Hint: consider a continuous, strictly increasing, and bounded function $f : [0, \infty) \to [0, \infty)$.

(c) Show that the linear space generated by the functions of the form e^{-nx^2}, $n = 1, 2, 3 \ldots$, is dense (with respect to sup-norm) in the space of all continuous functions $f : [0, \infty] \to \mathbb{R}$ having the property that $f(\infty) = 0$.

Proof. For context on this space, review the information surrounding Proposition 4.36 in Folland.

(a) Take an open cover $\mathcal{U}$ of $[0, \infty]$. There is some $U \in \mathcal{U}$ containing ∞; then $\mathcal{U}$ covers U^c, which is compact, and the finite subcover $(U_i)_1^n$ of U^c together with U covers $[0, \infty]$.

(b) We construct a homeomorphism $\tilde{f} : [0, \infty] \to [0, 1]$. Since $[0, 1]$ is metrizable, this will complete the proof.

Let $f : [0, \infty) \to [0, 1)$ be a continuous increasing bijection (such as $f(x) = \frac{2}{\pi} \tan^{-1}(x)$). Then there is a continuous extension $\tilde{f}(\infty) = 1$ iff $f - 1 \in C_0([0, \infty))$. But this is clear: since f is increasing, $\{x : |f(x)| \geq \varepsilon\} = [0, \tan(\frac{\pi}{2}(1 - \varepsilon))]$. Similarly, $\tilde{f}^{-1}$ is continuous since $[0, \infty]$ is compact and $[0, 1]$ is Hausdorff.

(c) You already know this is Stone-Weierstrass. It is your favorite theorem, how could you not know?

Let E be the span of these functions. It is easy to see this is an algebra. Also, $f(x) = e^{-x^2}$ separates points of $[0, \infty]$ since it is decreasing, and $f(\infty) = \lim_{n\to\infty} e^{-n^2} = 0$. Stone-Weierstrass completes the proof. □

10 August 2019

Problem 1. *Let $(X, \mathcal{M}, \mu)$ be a measure space and f a measurable non-negative function on X. Define $\nu : \mathcal{M} \to [0, \infty]$ by*

$$\nu(E) = \int_E f d\mu.$$

(a) Prove that ν is a measure.

Proof. Indeed, it's clear that $\nu(E) = \int_E f d\mu \geq 0$ for all E since f is assumed to be non-negative. It's equally clear that $\nu(\emptyset) = \int_\emptyset f d\mu = 0$.

We are only left to prove countable additivity. Take a countable collection $\{E_i\}$ of pairwise disjoint sets in $\mathcal{M}$, so we see for finitely many

$$\nu\left(\bigcup_{k=1}^N E_k\right) = \int_{\bigcup_{k=1}^N E_k} f d\mu = \int \chi_{\bigcup_{k=1}^N E_k} f d\mu = \sum_{k=1}^N \int \chi_{E_k} f d\mu = \sum_{k=1}^N \int_{E_k} f d\mu = \sum_{k=1}^N \nu(E_k).$$

By the monotone convergence theorem (the finite sums of characteristic functions form an increasing sequence that converges to the infinite sum pointwise), then ν is countably additive. Hence, ν is a measure. □

(b) Prove that $g \in L^1(\nu)$ if and only if $gf \in L^1(\mu)$ and in that case $\int_X g d\nu = \int_X gf d\mu$.

Proof. First we show that $\nu \ll \mu$. Indeed, if $\mu(E) = 0$ then choose an increasing sequence of simple functions f_n such that $f_n \to f$. Then by monotone convergence theorem and the definition of integral for simple functions, we have

$$\nu(E) = \int_E f d\mu = \int_E (\lim f_n) d\mu = \lim \int_E f_n d\mu = 0.$$

Then we may apply Radon-Nikodym theorem to see that $f = \frac{d\nu}{d\mu}$ and see that $g \in L^1(\nu)$ if and only if $\int_X |g| d\nu < \infty$ which is equivalent to $\int_X |g| f d\mu = \int_X |g| \frac{d\nu}{d\mu} d\mu < \infty$. Since f is non-negative, this is equivalent to having $\int_X |gf| d\mu < \infty$. Radon-Nikodym also tells us that $\int_X g d\nu = \int_X gf d\mu$. □

Problem 2. *(a) State Fatou's lemma.*

Proof. For $f_n \in L^+$ then

$$\int \liminf f_n \leq \liminf \int f_n$$

□

(b) State the dominated convergence theorem

Proof. Let $g, g_n \in L^+$ be measurable, $|f_n| \le g_n$ μ-a.e., $f_n \to f$ and $g_n \to f$ μ-a.e. with $\int g_n \to \int g < \infty$. Then $\int f_n \to \int f$. Moreover, $\int |f - f_n| \to 0$. □

(c) *Let f_n, g_n, h_n, f, g, h be measurable functions on $\mathbb{R}^n$ satisfying $f_n \le g_n \le h_n$, $f_n \to f$ a.e., $g_n \to g$ a.e., and $h_n \to h$ a.e. Suppose moreover that $f, h \in L^1$ and $\int f_n \to \int f$, $\int h_n \to \int h$. Prove that $g \in L^1$ and $\int g_n \to \int g$.*

Proof. We know $g_n - f_n \ge 0$ and $h_n - g_n \ge 0$. By Fatou's Lemma,

$$\int g - \int f = \int \liminf (g_n - f_n) \le \liminf \int (g_n - f_n) = \liminf \int g_n - \int f.$$

(A quick note on the last equality: usually $\liminf(\int g_n - \int f_n) \ge \liminf \int g_n - \int f$, but any subsequential limit of $\int f_n$ is still $\int f$, so we get equality here.)

Since $\int f < \infty$, we get $\int g \le \liminf \int g_n$.

Similarly,

$$\int h - \int g = \int \liminf (h_n g_n) \le \liminf \int (h_n - g_n) = \int h - \limsup \int g_n,$$

so $\int h < \infty$ implies $\limsup \int g_n \le \int g$. So $\limsup \int g_n \le \int g \le \liminf \int g_n$ and $\int g_n \to \int g$.

We have $|g| \le |h|$ whenever $g \ge 0$ and $|g| \le |f|$ whenever $g < 0$. So $|g| \le |f| + |h| \in L^1$. □

Problem 3. *Let $\{A_k\}_{k=1}^\infty$ be measurable subsets of a measure space and define B_m to be the set of all points which are contained in at least m of the sets $\{A_k\}_{k=1}^\infty$. Prove that B_m is measurable and*

$$\mu(B_m) \le \frac{1}{m} \sum_{k=1}^\infty \mu(A_k).$$

Proof. Let $C = \{F \subseteq \mathbb{N} \mid |F| = m\}$ which is a countable infinite set. Then we may express

$$B_m = \bigcup_{F \in C} \bigcap_{i \in F} A_i.$$

Therefore, each B_m is measurable.

Then

$$\chi_{B_m}(x) = 1 \iff \sum_{k=1}^\infty \chi_{A_k}(x) \ge m,$$

so thus $m\chi_{B_m} \le \sum_{k=1}^\infty \chi_{A_k}$. Therefore

$$m\mu(B_m) = \int m\chi_{B_m}\, d\mu \le \sum_{k=1}^\infty \int \chi_{A_k}\, d\mu = \int_{k=1}^\infty \mu(A_k).$$

□

Problem 4. *Let E be a subset of $\mathbb{R}$ which is not Lebesgue measurable. Prove that there exists an $\eta > 0$ such that for any two Lebesgue measurable sets A, B satisfying $A \subseteq E \subseteq B$ one has $\lambda(B\backslash A) > \eta$, where λ denotes Lebesgue measure.*

Proof. Since E is nonmeasurable, there exists $z \in \mathbb{Z}$ such that $E_z := E \cap [z, z+1)$ is nonmeasurable. So suppose we can construct sequences (A_n), (B_n) s.t. $A_n, B_n \subseteq [z, z+1)$,

$$A_n \subseteq A_{n+1} \subseteq E_z \subseteq B_{n+1} \subseteq B_n \quad \forall n,$$

and $\lambda(B_n\backslash A_n) < \frac{1}{n}$. Take $A = \bigcup A_n$ and $B = \bigcap B_n$; then $\lambda(B\backslash A) = 0$ and $E_z\backslash A \subseteq B\backslash A$. But λ is a complete measure, so $E_z\backslash A$ is measurable, as is $A \cup (E_z\backslash A)$, contradiction. □

Problem 5. *Let $\{A_k\}_{k=1}^{\infty}$ be Lebesgue measurable sets in $\mathbb{R}^n$ equipped with Lebesgue measure λ.*

(a) Prove that if $A_k \subseteq A_{k+1}$ for all k then $\lambda(\bigcup_{k=1}^{\infty} A_k) = \lim_{k\to\infty} \lambda(A_k)$

Proof. We will assume that λ is subadditive, so $\lambda(\bigcup_1^{\infty} A_k) \leq \sum_1^{\infty} \lambda(A_k)$.Then by setting $A_k = \emptyset$, we have

$$\lambda\left(\bigcup_1^{\infty} A_k\right) = \sum_1^{\infty} \lambda(A_j\backslash A_{j-1}) = \lim_{n\to\infty} \sum_1^{n} \lambda(A_j\backslash A_{j-1}) = \lim_{n\to\infty} \lambda(A_n).$$

□

(b) Prove that if $A_{k+1} \subseteq A_k$ for all k and $\lambda(A_1) < \infty$ then $\lambda(\bigcap_{k=1}^{\infty} A_k) = \lim_{k\to\infty} \lambda(A_k)$

Proof. Let $B_j = A_1\backslash A_j$ so $B_1 \subseteq B_2 \subseteq \ldots$, and $\lambda(A_1) = \lambda(B_j) + \lambda(A_j)$, and $\bigcup_1^{\infty} B_j = E_1\backslash(\bigcap_1^{\infty} A_j)$. Then by part (a), we have

$$\lambda(A_1) = \lambda\left(\bigcap_1^{\infty} A_j\right) + \lim_{j\to\infty} \lambda(B_j) = \lambda\left(\bigcap_1^{\infty} A_j\right) + \lim_{j\to\infty} (\lambda(A_1) - \lambda(A_j)).$$

Since $\lambda(A_1) < \infty$, we may subtract it from both sides to yield the desired result. □

(c) Give an example showing that without assuming $\lambda(A_1) < \infty$ the conclusion of the previous part does not hold.

Proof. Consider $A_j = [j, \infty)$ so that for each j, $\lambda(A_j) = \infty$ but $\bigcap_1^{\infty} A_j = \emptyset$ so $\lambda(\bigcap_1^{\infty} A_j) = 0$. □

Problem 6. *Let X and Y be Banach spaces. Show that the linear space $X \oplus Y$ is a Banach space under the norm $\|(x, y)\| = \|x\| + \|y\|$. Also determine (with justification) the dual $(X \oplus Y)^*$.*

Proof. A bit of a pencil pusher :)

First, $\|(x, y)\|$ is a norm since

(i) $\|x\| + \|y\| = 0 \iff \|x\|, \|y\| = 0 \iff x, y = 0 \iff (x, y) = 0.$

(ii) $\|(x_1,y_1)+(x_2,y_2)\| = \|x_1+x_2\| + \|y_1+y_2\| \leq \|x_1\|+\|y_1\|+\|x_2\|+\|y_2\|$.

(iii) $\|\lambda(x,y)\| = \|(\lambda x, \lambda y)\| = |\lambda|(\|x\|+\|y\|) = |\lambda|\|(x,y)\|$.

Suppose $\sum\|(x_i,y_i)\| < \infty$. Then so is $\sum\|x_i\|$, $\sum\|y_i\|$, so there exist $x \in X$, $y \in Y$ such that $\sum x_n \to x$, $\sum y_n \to y$ in norm. So

$$\begin{aligned}\|(x,y) - \sum_{i=1}^n (x_n,y_n)\| &= \|(x - \sum_{i=1}^n x_i, y - \sum_{i=1}^n y_i\| \\ &= \|x - \sum_{i=1}^n x_i\| + \|y - \sum_{i=1}^n y_i\| \to 0.\end{aligned}$$

We claim $(X\oplus Y)^* = X^*\oplus Y^*$, where $\phi\oplus\psi(x,y) = \phi(x)+\psi(y)$. First, given $\phi\in X^*$, $\psi\in Y^*$, we claim $\phi\oplus\psi$ is linear:

$$\begin{aligned}(\phi\oplus\psi)(\lambda x_1+x_2),\lambda y_1+y_2) &= \lambda\phi(x_1)+\lambda\psi(y_1)+\phi(x_2)+\psi(y_2) \\ &= \lambda(\phi\oplus\psi)(x_1,y_2)+(\phi\oplus\psi)(x+2,y+2).\end{aligned}$$

It is also clear $\|\phi\oplus\psi\| \leq \|\phi\|+\|\psi\|$, so this is bounded.

Now let $\xi \in (X\oplus Y)^*$. Note

$$\xi(x,y) = \xi(x,0)+\xi(0,y),$$

and let $\xi_X \in X^*$, $\xi_Y \in Y^*$ be

$$\xi_X x = \xi(x,0) \quad \text{and} \quad \xi_Y y = \xi(0,y).$$

Then $\|\xi_X\| = \sup_{\|x\|=1}\|\xi(x,0)\| \leq \|\xi\| \geq \|\xi_Y\|$. We still need to show the norm is preserved between these two spaces: i.e., $\|\phi\oplus\psi\| = \|\phi\|+\|\psi\|$. But all we need is some sequences $(x_n)\subset X$ and $(y_n)\subset Y$ approximating the norm of ϕ,ψ in these respective spaces; then $\|(\phi\oplus\psi)(x_n,y_n)\| = \|\phi(x_n)\|+\|\psi(y_n)\|$ approaches this sum. So $\xi = \xi_X\oplus\xi_Y$. □

Problem 7. *For each $n\in\mathbb{N}$ define on ℓ^∞ the linear functional $\varphi_n(x) = n^{-1}\sum_{k=1}^n x(k)$. Let φ be the weak* cluster point of the sequence $\{\varphi_n\}$. Show that φ does not belong to the image of ℓ^1 under the canonical embedding $\ell^1 \hookrightarrow (\ell^\infty)^*$.*

Proof. See August 2018, Problem 10(c). □

Problem 8. *Let $T: X\to Y$ be a surjective linear map between Banach spaces and suppose that there is a $\lambda>0$ such that $\|Tx\|\geq\lambda\|x\|$ for all $x\in X$. Show that T is bounded.*

Proof. See January 2009, Problem 6, amongst others. □

Problem 9. *Let X be a compact metric space and μ a regular Borel measure on X. Let $f: X\to [0,\infty)$ be a continuous function and for each $n\in\mathbb{N}$ set $f_n(x) = f(x)^{1/n}$ for all $x\in X$. Show that $\int f_n d\mu \to \mu(\operatorname{supp} f)$ as $n\to\infty$ where $\operatorname{supp} f = \{x\in X \mid f(x)>0\}$.*

Proof. See January 2011, Problem 4, amongst others. □

Problem 10. *Let X be a compact metric space and let $x \in X$. Suppose that the point mass δ_x is the weak* limit of a sequence of atomless Radon measures on X (viewing all of these measures as elements of $C(X)^*$). Show that every neighborhood of x is uncountable.*

Proof. Recall: if $\mu_n \to \delta_x$ weak*, this means for all $f \in C(X)$, $\int f\,d\mu_n \to \int f\,d\delta_x$. Suppose $U \ni x$ open such that U is countable. By Urysohn, $\exists f \in C(X)$ such that $f(x) = 1$, $0 \le f \le 1$, and $f = 0$ on U^c. Then

$$1 = f(x) = \lim_n \int f\,d\mu_n \le \limsup_n \int \chi_U\,d\mu_n = \limsup_n \mu_n(U).$$

Thus there is some n such that $\mu_n(U) > 0$. Write $U = \{u_1, u_2, \dots\}$. Then since $\sum_{k=1}^{\infty} \mu_n(\{u_k\}) = \mu_n(U) > 0$, there must be some k such that $\mu_n(\{u_k\}) > 0$. So $\{u_k\}$ is an atom for μ_n which contradicts the fact that μ_k is atomless. ☐

11 January 2019

Problem 1. *True or false (prove or give a counter example)*

(a) Let $E \subseteq \mathbb{R}$ be a Borel set, then $\{(x,y) \in \mathbb{R}^2 \mid x - y \in E\}$ is a Borel set in $\mathbb{R}^2$.

Proof. TRUE.

Define $f(x,y) = x - y : \mathbb{R}^2 \to \mathbb{R}$. This is continuous. Let

$$\mathcal{A} := \{S \subseteq \mathbb{R} \mid f^{-1}(S) \text{ is a Borel set of } \mathbb{R}^2\}$$

Then $\mathcal{A}$ is a σ-algebra (easy to check). If S is open, then $f^{-1}(S)$ is open in $\mathbb{R}^2$, thus Borel. So $\{\text{open sets}\} \subseteq \mathcal{A}$ and so the Borel algebra is a subset of $\mathcal{A}$. In particular, $E \in \mathcal{A}$. ☐

(b) Let $E \subseteq Q := [0,1] \times [0,1]$. Assume that for every $x, y \in [0,1]$ the sets $E_x = \{y \in [0,1] \mid (x,y) \in E\}$ and $E^y = \{x \in [0,1] \mid (x,y) \in E\}$ are Borel. Then E is Borel.

Proof. FALSE.

Consider a non-Borel set $A \subset [0,1]$. Set $E = \{(x,x) \mid x \in A\}$. Then each E^y and E_x is a singleton which is Borel, but E is not. ☐

(c) A function $f : \mathbb{R} \to \mathbb{R}$ is called Lipschitz if there exits a $M > 0$ such that $\forall x, y \in \mathbb{R}$, $|f(x) - f(y)| \le M|x - y|$. If $A \subseteq \mathbb{R}$ is Lebesgue measureable and f is Lipschitz then $f(A)$ is Lebesgue measurable.

Proof. TRUE.

Since A is Lebesgue measurable, then we can write $A = \left(\bigcup_j K_j\right) \cup N$ where each K_j is a compact set and N has Lebesgue measure zero. Then $f(A) = \left(\bigcup_j f(K_j)\right) \cup f(N)$. It's clear that each $f(K_j)$ is Lebesgue measurable, since f is Lipschitz. We are only left to see that $f(N)$ is also Lebesgue measurable.

Indeed, for every $\epsilon > 0$ we can write $N \subseteq \bigcup_k B_k$ where each B_k is a ball of radius r_k and $\sum_k m(B_k) < \epsilon$. But then by Lipschitz continuity, $f(B_k)$ is contained in a ball of radius Mr_k where M is the Lipschitz constant of f. Thus, $m(f(B_k)) \leq Mm(B_k)$ so that $m(f(N)) \leq M\sum_k m(B_k) < M\epsilon$. Let $\epsilon \to 0$ so $f(N)$ must have outer measure equal to zero, hence it is a null set. □

Problem 2. *Let $(X, \mathcal{F}, \mu)$ be a measure space. is it true that for every measurable essentially bounded $f : X \to \mathbb{R}$ we have $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$? Give an answer both in the case that μ is finite and the case that μ is σ-finite.*

Proof. If μ is finite: By Hölder, we know that $\|f\|_p \leq \|f\|_q$ when $p \leq q$. Also, $\|f\|_p \leq \|f\|_\infty$ for all p. Therefore, $\|f\|_p \nearrow\leq \|f\|_\infty$ and so $\lim_p \|f\|_p \leq \|f\|_\infty$.

On the other hand, for every $\epsilon > 0$, let $E = \{x \mid |f(x)| > \|f\|_\infty - \epsilon\}$ and $0 < \mu(E) \leq 1$ since $\|f\|_\infty = \operatorname{esssup}|f(x)| < \infty$. Then $\|f\|_p^p \geq \int_E |f|^p > \left(\|f\|_\infty - \epsilon\right)^p \mu(E)$. Take $p \to \infty$ so $\lim_p \|f\|_p \geq \|f\|_\infty - \epsilon$, implying $\lim_p \|f\|_p \geq \|f\|_\infty$.

If μ is σ-finite: No, this is not true. Consider $f(x) = \frac{1}{x}$ on $[1, \infty)$. Then $\lim_p \|f\|_p = 0 \neq \|f\|_\infty = 1$. □

Problem 3. *Let $f : \mathbb{R} \to \mathbb{R}$ Lebesgue integrable and for $n \in \mathbb{N}$ define*

$$g_n(x) = n \int_{(x,x+1/n)} f d\lambda.$$

(a) Prove that $\lim_{n\to\infty} g_n = f$ λ-a.e.

Proof. This is Lebesgue Differentiation Theorem, with $E_r = (x, x + r)$. □

(b) Prove that for every $n \in \mathbb{N}$, $\int_\mathbb{R} |g_n| d\lambda \leq \int_\mathbb{R} |f| d\lambda$.

Proof.

$$\begin{aligned}
\int_\mathbb{R} |g_n(y)|\, d\lambda(y) &= \int_\mathbb{R} |n \int_y^{y+\frac{1}{n}} f(t)\, d\lambda(t)|\, d\lambda(y) \\
&\leq n \int_\mathbb{R} \int_y^{y+\frac{1}{n}} |f(t)|\, d\lambda(t)\, d\lambda(y) \\
&\overset{\text{Tonelli}}{=} n \int_\mathbb{R} \int_{t-\frac{1}{n}}^{t} |f(t)|\, d\lambda(y)\, d\lambda(t) \\
&= n \int_\mathbb{R} \frac{1}{n} |f(t)|\, d\lambda(t) \\
&= \int_\mathbb{R} |f|\, d\lambda.
\end{aligned}$$

□

(c) Prove $\lim_{n\to\infty} \int_\mathbb{R} |g_n| d\lambda = \int_\mathbb{R} |f| d\lambda$.

Proof. Apply dominated convergence theorem with parts (a) and (b). □

Problem 4. *Let $f \in L^1((0,1]^2, \lambda_2)$ such that $\int_{(0,x]\times(0,y]} f d\lambda_2 = 0$ for every $x, y \in (0,1]$. Prove that $f = 0$ λ_2-a.e.*

Proof. First note that

$$(a,b)\times(c,d) = \bigcup_n ((0, b-1/n]\times(0, d-1/n]) \setminus ((0,a]\times(0,1] \cup (0,1]\times(0,b])$$

And since all open rectangles generate all Borel sets in $\mathbb{R}^2$, then we have that for every Borel set $B \subseteq \mathbb{R}^2$, $\int_B f d\lambda_2 = 0$.

Since every Lebesgue set A is of the form $A = B \cup N$ where B is a Borel measurable set and N is a set of measure zero. Hence, $\int_A f d\lambda_2 = 0$ for any Lebesgue measurable set A.

Now consider $A^+ = \{x \mid f(x) > 0\}$ and $A_- = \{x \mid f(x) < 0\}$. Since both are measurable, then $\int_{A^+} f d\lambda_2 = 0 = \int_{A^-} f d\lambda_2$. Hence, $f = 0$ λ_2-a.e. □

Problem 5. *Let λ be the Lebesgue measure on $\mathbb{R}$. Let $E \subseteq \mathbb{R}$ be Lebesgue measurable such that $0 < \lambda(E) < \infty$. Prove that for all $0 \le \gamma < 1$ there exists an open interval $I \subseteq \mathbb{R}$ such that*

$$\lambda(E \cap I) \ge \gamma\lambda(I).$$

Proof. Choose an open set $O \supset E$ such that $\lambda(E) \ge \gamma\lambda(O)$. We can write $O = \bigcup_i O_I$ for open and disjoint intervals O_i. Hence

$$E = E \cap O = E \cap \bigcup_i O_i = \bigcup_i (E \cap O_i)$$

Suppose to the contrary that $\lambda(E \cap O_i) < \gamma\lambda(O_i)$ for all i. Then

$$\lambda(E) = \lambda\left(\bigcup (E \cap O_i)\right) = \sum_i \lambda(E \cap O_i) < \gamma \sum_i \lambda(O_i) = \gamma\lambda(O)$$

which is a contradiction with the fact that $\lambda(E) \ge \gamma\lambda(O)$. Hence, it must be that for some k, $\lambda(E \cap O_k) \ge \gamma\lambda(O_k)$. □

Problem 6. *Let X be a compact metrizable space and $\{\mu_n\}$ a sequence of Borel measures on X with $\mu_n(X) = 1$ for every n. Consider the linear map $\varphi : C(X) \to \ell^\infty(\mathbb{N})$ defined by $\varphi(f) = \left(\int_X f d\mu_n\right)_n$. What conditions on the sequence $\{\mu_n\}$ are equivalent to φ being an isometry? Provide justification.*

Proof. We claim ϕ is an isometry iff $\forall U \subseteq X$ open, nonempty,$\sup_n \mu_n(U) = 1$.

($\Leftarrow$) If there exists such a set such that $\mu_n(U) < 1-\varepsilon$ for all n for some $\varepsilon > 0$, then by Urysohn (metrizable$\Rightarrow$Hausdorff) there is a continuous function $f : X \to [0,1]$ such that f is 0 on U^c. Then

$$|\int f\,d\mu_n| = |\int_U f\,d\mu_n \leq \mu_n(U) < 1-\varepsilon.$$

So $\|\phi(f)\|_\infty \leq 1-\varepsilon < 1 = \|f\|_{C(X)}$.

($\Rightarrow$) Fix $\varepsilon > 0$ and consider

$$U := \{x \in X : |f(x)| > \|f\|_{C(x)} - \varepsilon\}.$$

Then U is nonempty and open in X. So

$$\begin{aligned}|\int f\,d\mu_n| &= |\int_U f\,d\mu_n + \int_{U^c} f\,d\mu_n| \\ &\geq |\int_U f\,d\mu_n| - |\int_{U^c} f\,d\mu_n| \\ &\geq (\|f\|_{C(X)} - \varepsilon)\mu_n(U) - \|f\|_{C(X)}\mu_n(U^c).\end{aligned}$$

So

$$\sup_n |\int f\,d\mu_n| \geq \|f\|_{C(X)} - \varepsilon,$$

and hence $\|\phi(f)\|_\infty \geq \|f\|_{C(X)}$. □

Problem 7. *Let X be a compact metric space and $\{f_n\}$ a sequence in $C(X)$. Prove that $\{f_n\}$ converges weakly in $C(X)$ if and only if it converges pointwise and $\sup_n \|f_n\| < \infty$. Also, give an example of an X and a sequence $\{f_n\}$ in $C(X)$ which converges weakly but not uniformly.*

Proof. By considering $f_n - f$, we may assume without loss of generality that f_n converges to 0.

$\Rightarrow$) We know $C[0,1]^* = \mathcal{M}[0,1]$. Then $f_n \to 0$ weakly implies $\int f_n d\mu \to 0$ for all $\mu \in \mathcal{M}[0,1]$. Choose $\mu = \delta_t$ so

$$\int f_n d\delta_t = f_n(t) \to 0 \quad \forall t \in [0,1]$$

(this follows from the fact that weak convergence implies uniformly bounded). Consider

$$\begin{aligned}\chi &: C[0,1] \to C[0,1]^{**} = \mathcal{M}[0,1]^* \\ \chi(f_n)(\mu) &= \mu(f_n)\end{aligned}$$

Since $\mu(f_n) \to 0$ then $\chi(f_n)(\mu) \to 0$ for all $\mu \in \mathcal{M}[0,1]$. Since convergent sequences are bounded, then $\sup_n |\chi(f_n)(\mu)| \leq M$.

By the uniform boundedness theorem, $\sup_n \|\chi(f_n)\| < \infty$. By isometry, $\|f_n\| = \|\chi(f_n)\|$ so $\sup_n \|f_n\| < \infty$.

$\Leftarrow$) By Dominated Convergence Theorem, $f_n \to 0$ in $L^1(\mu)$. So therefore, $|\int f_n d\mu| \le \int |f_n| d|\mu| \to 0$. So $f_n \to 0$ weakly.

Example: Take $X = [0,1]$ and consider the functions

$$f_n(x) = \begin{cases} nx & x \in [0, 1/n] \\ -nx + 2 & x \in (1/n, 2/n] \\ 0 & x \in (2/n, 1] \end{cases}$$

Then they converge to 0 weakly, but not strongly. □

Problem 8. *Let X be a Banach space. Show that if X^{**} is separable then so is X. Also, give an example, with justification, to show that the converse is false.*

Proof. We will show the weaker result that states that if the dual X^* is separable, then so is X.

Let X^* be separable. Consider the unit sphere $S_{X^*} = \{\varphi \in X^* \mid \|\varphi\| = 1\}$. Then S_{X^*} is separable and so we can let $\{\varphi_n\}$ be a countable dense subset of S_{X^*}.

For each $n \in \mathbb{N}$, choose $x_n \in \mathbb{N}$ with $\|x_n\| = 1$ such that $|\varphi_n(x_n)| > 1/2$. Let $D = \overline{\text{span}}\{x_1, x_2, \ldots\}$.

Then D is countable; ex. we can consider the following set countable and dense subset of D:

$$\bigcup_{n \in \mathbb{N}} \left\{ \sum_{j=1}^{n} (a_j + ib_j) x_j \mid a_j, b_j \in \mathbb{Q} \right\}$$

We want to show that $D = X$. Suppose it were not, then there ewould be some $\varphi \in S_{X^*}$ with $\varphi|_D = 0$. Since $\{\varphi_n\}$ is dense, there exists some n such that $\|\varphi - \varphi_n\| < 1/4$. Therefore,

$$\frac{1}{2} \le |\varphi_n(x_n)| = |\varphi_n(x_n) - \varphi(x_n)| \le \|\varphi_n - \varphi\| \|x_n\| < \frac{1}{4}.$$

This is a contradiction and hence, $D = X$.

example. c_0 is separable, but $\ell^\infty = c_0^{**}$ is not separable. □

Problem 9. *(a) Let X be a compact metrizable space. Describe the dual of $C(X)$ according to the Riesz representation theorem.*

Proof. For every $\varphi \in C(X)^*$, there exists a unique finite regular signed measure μ on the Borel subsets of X such that

$$\varphi(f) = \int_X f d\mu$$

for each $f \in C(X)$. Moreover, $\|\varphi\| = |\mu|(X)$.

□

(b) *Consider the spaces $X = \{1/n \mid n \in \mathbb{N}\} \cup \{0\}$ and $Y = [0,1]$ with the topologies inherited from $\mathbb{R}$. Prove that there does not exist a bijective bounded linear map from $C(X)$ to $C(Y)$.*

Proof. By contradiction. Suppose there exists a bijective bounded linear map $T : C(X) \to C(Y)$. Then by the Open Mapping Theorem (or more accurately, the corollary that is the Bounded Inverse Theorem), then T^{-1} is a bijective bounded linear map from $C(Y)$ to $C(X)$. This says that the two spaces are isomorphic.

Therefore, the transpose map induces an isomorphism from $C(X)^* \cong C(Y)^*$. I.e., $T^{\perp}(f)(y) = f(Ty)$ has norm $\|T\|$ and has a bounded inverse $(T^{\perp})^{-1}(g)(x) = g(T^{-1}x)$. We claim $C(X)^*$ is separable while $C(Y)^*$ is not.

The point-masses on Y are each of distance 2 from each other, since continuous functions separate points. Since Y has uncountably many points, $(B(1, \delta_y))_{y \in Y} \subset M(Y)$ is an uncountable disjoint collection of open sets, showing $C(Y)^*$ is not separable.

We claim the rational span of $(\delta_{1/n})_n$ is dense in $C(X)^*$. Any measure on $X \cap (0,1]$ can be approximated uniformly by rational point-mass functions as $X \cap (0,1]$ is totally disconnected (for example, given $\mu \in M(X)$ and for each $\frac{1}{n}$ we can choose q_n^i such that $|\mu(\frac{1}{n}) - q_n^i| \leq \frac{1}{2^{-n}}$). Now $\mu(X) = \lim \mu(\bigcup_{k=1}^{\infty} X \cap [1/k, 1])$ since μ is Radon, so $\mu(\{0\}) = 0$ and the proof is complete. □

Problem 10. *Let X be a Banach space and Y a subspace of X. Show that $\|x + Y\| = \inf\{\|x + y\| \mid y \in Y\}$ defines a norm on X/Y if and only if Y is closed.*

Proof. $\Leftarrow$) Suppose Y is closed. It's easy to see $\|x + Y\|$ is well-defined and a semi-norm. Suppose $\|x + Y\| = 0$. Then there exists $y_n \in Y$ such that $\|x - y_n\| \to 0$. Since Y is closed, then $x \in Y$. Therefore, $x + Y = Y = 0 + Y$ which is the zero vector in X/Y.

$\Rightarrow$) Suppose this is a norm. Take any convergent sequence y_n in Y with $y_n \to y'$. Then $\inf_{y \in Y} \|y - y'\| \leq \|y_n - y'\| \to 0$ and so $\|y' + Y\| = 0$. Since this is a norm, then $y' + Y = 0 + Y = Y$ and so $y' \in Y$. Hence Y must be closed. □

12 August 2018

(Solve any 10 of the following 12 problems)

Problem 1. *Let μ and ν be positive measures on the same measurable space with ν finite and absolutely continuous with respect to μ. Show that for every $\epsilon > 0$, there exists $\delta > 0$ such that $\mu(E) < \delta$ implies $\nu(E) < \epsilon$.*

Proof. Suppose for contradiction that $\exists \epsilon > 0$ such that $\mu(E) < \delta$ then $\nu(E) \geq \epsilon$ for all $\delta > 0$ adn for some E. We'll construct the set E_n to be some set with $\mu(E_n) < 2^{-n}$. Let $F_k = \bigcup_{n=k}^{\infty} E_n$ so $\mu(F_k) < 2^{-k+1}$.

Let $F = \bigcap_{k=1}^{\infty} F_k$ so $\mu(F) = 0$. Since $\nu \ll \mu$, then $\nu(F) = 0$.

However, since F_k is a decreasing sequence, we have

$$\nu(F) = \lim_n \nu\left(\bigcap_{k=1}^n F_k\right) = \lim_n \nu(F_n) \geq \epsilon.$$

Contradiction! □

Problem 2. *Let μ be a positive measure. Suppose that $(f_n)_{n=1}^\infty$ is a Cauchy sequence in $L^1(\mu)$. Show that for all $\epsilon > 0$ there exists a $\delta > 0$ such that $\mu(E) < \delta$ implies*

$$\forall n \geq 1 \qquad \left|\int_E f_n d\mu\right| < \epsilon.$$

You may use without proof the result of problem #1.

Proof. Let $\epsilon > 0$. Since $\{f_n\}$ is Cauchy in $L^1(\mu)$, there exists $f \in L^1(mu)$ such that $f_n \to f$ in $L^1(\mu)$ as $n \to \infty$, since $L^1(\mu)$ is a Banach space.

Define $\nu(E) := \left|\int_E f d\mu\right|$.

Then by Problem 1, there exists some $\delta > 0$ such that $\nu(E) = \left|\int_E f d\mu\right| < \epsilon/2$ when $\mu(E) < \delta$, thenf or large enough n (say $n \geq N$) we have

$$\left|\int_E f_n d\mu\right| = \left|\int_E (f_n - f + f) d\mu\right| \leq \left|\int_E f_n - f d\mu\right| + \left|\int_E f d\mu\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for $\mu(E) < \delta$.

For each i, N, we can find δ_i such that $\left|\int_E f_i d\mu\right| < \epsilon$ when $\mu(E) < \delta_i$. By the same reasoning as above, if we set $\tilde{\delta} = \min\{\delta_1, \ldots, \delta_{N-1}, \delta\}$ then $\left|\int_E f_n d\mu\right| < \epsilon$ whenever $\mu(E) < \tilde{\delta}$ for all $n \in \mathbb{N}$. □

Problem 3. *Let $f : [0,1] \to [0,\infty)$ be Lebesgue measurable. For $n \in \mathbb{N}$ define*

$$g_n = \frac{f^n}{1 + f^n}.$$

(a) *Explain why $\int_0^1 g_n(t)dt$ exists and is finite for all n.*

   Proof. Since $g_n = \frac{f^n}{1+f^n} \leq 1$ for all n, then $\int_0^1 g_n dx \leq \int_0^1 1 dx = 1$ for all n. □

(b) *Prove that $\lim_n \int_0^1 g_n(t)dt$ exists and find an expression for it. Make sure to state which major theorems you are using in your proof.*

   Proof. Define $E_1 = \{x \mid 0 \leq f(x) < 1\}$, $E_2 = \{x \mid f(x) = 1\}$ and $E_3 = \{x \mid f(x) > 1\}$.
   If $x \in E_1$ then $g_n(x) = \frac{f^n(x)}{1+f^n(x)} \to 0$. So by DCT, $\lim_n \int_{E_1} g_n dx = \int_{E_1} 0 dx = 0$.
   If $x \in E_2$ then $g_n(x) = \frac{f^n(x)}{1+f^n(x)} = \frac{1}{2}$ for all n and so

$$\lim_n \int_{E_2} g_n dx = \int_{E_2} \frac{1}{2} dx = \frac{1}{2} m(E_2).$$

If $x \in E_3$ then $g_n(x) = \frac{f^n(x)}{1+f^n(x)} \to 1$ and so by DCT,

$$\lim_n \int_{E_3} g_n dx = \int_{E_3} dx = m(E_3).$$

Thus,

$$\lim_n \int_0^1 g_n dx = \lim_n \int_{E_1} g_n dx + \int_{E_2} g_n dx + \int_{E_3} g_n dx = \frac{1}{2} m(E_2) + m(E_3).$$

□

Problem 4. *Consider $C([0,1])$ endowed with its usual uniform norm. Prove or disprove that there is a bounded linear functional φ on $C([0,1])$ such that for all polynomials p, we have $\varphi(p) = p'(0)$, where p' is the derivative of p.*

Proof. DISPROVE.

Consider $p_n = 1 - (x-1)^n$ so then $\|p_n\|_\infty = 1$ but $p_n'(0) = n \to \infty$. If such a φ existed, then $n = |\varphi(p_n)| = \|\varphi(p_n)\| \leq c\|p_n\|$ which cannot happen. □

Problem 5. *(a) Define the product topology on the Cartesian product $\Pi_{\alpha \in A} X_\alpha$ of a family of topological spaces $(X_\alpha)_{\alpha \in A}$*

Proof. The product topology is the weak topology generated by $\pi_\alpha : \prod_{\alpha \in A} X_\alpha$ being the coordinate maps. Its subbase is the collection $\pi_\alpha^{-1}(U_\alpha)$ for U_α open in X_α. □

(b) State Tychonoff's compactness theorem.

Proof. If $\{X_\alpha\}$ is a family of compact topological spaces then $\Pi_{\alpha \in A} X_\alpha$ is compact. □

(c) State and prove the Banach-Alaoglu theorem (Hint: Use Tychonoff's theorem)

Proof. Thoerem: Let X be a normed vector space. The closed unit ball $\{f \in X^* \mid \|f\| \leq 1\}$ is compact in the weak*-topology.

For all $x \in X$, let $D_x := \{\xi \mid |\xi| \leq \|x\|\} \subseteq \mathbb{C}$. Then D_x is compact, and by Tychonoff's theorem, $D := \Pi_{x \in X} D_x$ is comapct. Define complex function φ with $\varphi(x) \leq \|x\|$.

We define $B^* \subseteq D$ to consist of linear functions of D. We claim B^* is closed. Indeed, let $\{f_\alpha\}$ be a net in B^* that converges to f. Then

$$f(ax+by) = \lim f_\alpha(ax+by) = \lim(af_\alpha(x) + bf_\alpha(y)) = a \lim f_\alpha(x) + b \lim f_\alpha(y) = af(x) + bf(y).$$

So $f \in B^*$. Since closed subsets of comapct spaces are compact, then B^* is compact in the weak*-topology. □

Problem 6. *Let (X, d) be a compact metric space.*

(a) Show that X has a countable, dense set $\{x_n \mid n \in \mathbb{N}\}$.

Proof. If X is countable, we are done. So suppose X is uncountable. Since X is compact, for all $n \in \mathbb{N}$, X can be covered by finitely many balls of radius $\frac{1}{n}$. For each n, choose such a finite cover with balls centered at the points $\{x_j^n\}_{j=1}^{N_n}$. Then the collection $E := \bigcup_n \{x_j^n\}_{j=1}^{N_n}$ is countable.

For $x \in X$, for all $n \in \mathbb{N}$, $x \in B(1/n, x_j^n)$ for some $x_j^n \in E$ so E is dense. □

(b) Let $f_n : X \to [0, \infty)$ be $f_n(x) = d(x, x_n)$. Show that if $x, y \in X$ and $f_n(x) = f_n(y)$ for all $n \in \mathbb{N}$, then $x = y$.

Proof. We then have that $d(x, x_n) = d(y, x_n)$ for all n. We know for all $m \in \mathbb{N}$ we can find x_m such that $d(x, x_m) < 1/m$ so $d(y, x_m) < 1/m$. So we can find a sequence $\{x_m\}_{m=1}^{\infty}$ such that $x_m \to x$ and $x_m \to y$ as $m \to \infty$. But X is a metric space and thus Hausdorff, so limits are unique. Therefore, $x = y$. □

Problem 7. *Let $K > 0$ and let Lip_K be the set of functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $|f(x) - f(y)| \leq K|x - y|$.*

(a) Prove that

$$d(f_1, f_2) = \sum_{j=0}^{\infty} 2^{-j} \sup_{x \in [-j,j]} |f_1(x) - f_2(x)|$$

defines a metric on Lip_K

Proof. First, suppose $d(f_1, f_2) = 0$. Then $\sum_{j=0}^{\infty} 2^{-j} \sup_{x\in[-j,j]} |f_1(x) - f_2(x)| = 0$ so $\sup_{x\in[-j,j]} |f_1(x) - f_2(x)| = 0$ for all j. Thus, $f_1(x) = f_2(x)$ for all x.

It's trivial to see that $d(f_1, f_2) = d(f_2, f_1)$.

Finally,we'll show the triangle inequality. This again follows directly: $|f_1(x) - f_2(x)| \leq |f_1(x) - f_3(x)| + |f_3(x) - f_2(x)|$ for all x. Taking sup on both sodies and multiplying by 2^{-j} we get $d(f_1, f_2) \leq d(f_1, f_3) + d(f_1, f_2)$. □

(b) Prove that Lip_K is a complete metric space

Proof. Suppose (f_n) is a Cauchy sequence in Lip_K. Then for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(f_1, f_m) = \sum_{j=1}^{\infty} 2^{-j} \sup_{x\in[-j,j]} |f_1(x) - f_2(x)| < \epsilon$. Then for each j and $x \in [-j, j]$ we have $|f_n(x) - f_m(x)| < \epsilon'$.

Thus, $\{f_n(\xi)\}$ is Cauchy sequence on $[-j, j]$ for each ξ. But we can find $f(x)$ such that $f_n(x) \to f(x)$.

We want to show that $d(f_n, f) \to 0$. Since $f_n(x) \to f(x)$, then for all $\epsilon > 0$ we can find some $N \in \mathbb{N}$ such that for all $n \geq N$, $|f_n(x) - f(x)| < \epsilon$. THen

$$d(f_n, f) = \sum_{j=1}^{\infty} 2^{-j} \sup_{x \in [-j,j]} |f_n(x) - f(x)| < \sum_{j=1}^{\infty} 2^{-j} \epsilon = \epsilon$$

So $d(f_n, f) \to 0$. To see $f \in \mathrm{Lip}_K$,

$$|f(x) - f(y)| = \left| \lim_n f_n(x) - \lim_n f_n(y) \right| = \lim_n |f_n(x) - f_n(y)| \leq K \lim_n |x - y| = K|x - y|.$$

□

Problem 8. *Let X, Y be topological spaces. A map $f : X \to Y$ is said to be proper if for every compact subset $K \subseteq Y$, the inverse image $f^{-1}(K)$ is compact.*

(a) Suppose X is a compact space and Y is Hausdorff. Prove that every continuous map $f : X \to Y$ is proper.

Proof. Let $K \subseteq Y$ be compact. Since Y is Hausdorff, then K is closed. Since f is continuous, and $Y \backslash K$ is open in Y then $f^{-1}(Y \backslash K)$ is open in X. So $f^{-1}(K) = X \backslash f^{-1}(Y \backslash K)$ is closed. Since X is compact, $f^{-1}(K)$ is compact. □

(b) Give an example of a continuous map which is not proper.

Proof. Consider the constant function $\mathbb{1} : \mathbb{R} \to \mathbb{R}$ which sends $x \mapsto 1$. So $\mathbb{1}^{-1}(\{1\}) = \mathbb{R}$. □

(c) Suppose $f : \mathbb{R}^m \to \mathbb{R}^n$ is a proper continuous map. Prove that f is a closed map, ie. $f(C)$ is closed in $\mathbb{R}^n$ whenever C is a closed subset of $\mathbb{R}^m$.

Proof. Let $\{y_n\} \subseteq f(C)$ with $y_n \to y$. Define $A = \{y\} \cup \{y_n\}$ (compact). Then $f^{-1}(A)$ is compact, so there exists $x_n \in f^{-1}(A) \cap C$ such that $f(x_n) = y_n$. Find a convergent subsequence x_{n_k} with $x_{n_k} \to x$ for $x \in C \cap f^{-1}(A)$. By continuity of f, we have $f(x) = y$. □

Problem 9. *Consider the interval $[-\pi, \pi]$ equipped with Lebesgue measure μ. For $n \in \mathbb{Z}$, consider the functions $f_n \in C([-\pi, \pi])$ given by $f_n(t) = e^{int}$.*

(a) Prove that $\mathrm{span}_{\mathbb{C}}\{f_n \mid n \in \mathbb{Z}\}$ is dense in the space

$$\mathcal{A} := \{f \in C([-\pi, \pi]) \mid f(-\pi) = f(\pi)\}$$

with respect to the uniform norm.

Proof. Let $\mathcal{B} = \mathrm{span}_{\mathbb{C}}\{f_n\} \subseteq \mathcal{A} \subseteq C([-\pi, \pi])$. Note that $\mathcal{B}$ separates points and is closed under complex conjugates. By Stone-Weierstrass, $\mathcal{B}$ is dense in $C[-\pi, \pi]$ hence also dense in $\mathcal{B}$. □

(b) Show that $\left\{ \frac{f_n}{\sqrt{2\pi}} \mid n \in \mathbb{Z} \right\}$ is an orthonormal basis for the Hilbert space $L^2([-\pi, \pi], \mu)$.

Proof. Note that

$$\|\langle f_n, f_n\rangle\|_2 = \left|\int_{-\pi}^{\pi} e^{int}e^{-int}dt\right|^{1/2} = \sqrt{2\pi}$$

For $n \neq m$,

$$\langle f_n, f_m\rangle = \int_{-\pi}^{\pi} e^{int}e^{-int}dt = \frac{e^{i(n-m)t}}{n-m}|_{-\pi}^{\pi} = 0.$$

So they are orthonormal. □

(c) *Is the following statement true or false?:*
"For every $f \in \mathcal{A}$, $f = \lim_{N\to\infty} \frac{1}{2\pi}\sum_{n=-N}^{N}\langle f, f_n\rangle f_n$ with respect to the uniform norm."
Give a brief explanation why or why not.

Proof. TRUE.
Claim: $\sum_{-\infty}^{\infty}\langle f, f_n\rangle f_n$ exists. By Pythagorean theorem, $\left\|\sum_{-\infty}^{\infty}\langle f, f_n\rangle f_n\right\| = \sum_{-\infty}^{\infty}\|\langle f, f_n\rangle f_n\|$. By Bessel's inequality, $\sum_{-\infty}^{\infty}\langle f, f_n\rangle f_n$ is bounded so it exists.
Let $g := f - \sum_{-\infty}^{\infty}\langle f, f_n\rangle f_n$ so that

$$\langle g, f_m\rangle = \langle f, f_m\rangle - \sum_{-\infty}^{\infty}\langle\langle f, f_n\rangle f_n, f_m\rangle = \langle f, f_m\rangle - \langle f, f_m\rangle = 0.$$

By completeness of Hilbert spaces, $g = 0$. So $f = \sum_{-\infty}^{\infty}\langle f, f_n\rangle f_n$. □

Problem 10. *Let $(X, \|\cdot\|)$ be a normed linear space and let $(X^*, \|\cdot\|_{X^*})$ denote its dual Banach space of bounded linear functionals. Recall that $\|\varphi\|_{X^*} = \sup_{\|x\|=1}|\varphi(x)|$ for $\varphi \in X^*$*

(a) *Prove that for each $x \in X$, there exits $\varphi \in X^*$ with $\|\varphi\|_{X^*} = 1$ and $\|x\| = \varphi(x)$.*

Proof. We will prove the more general case: let M be closed and $x \in X\backslash M$. Then there exists $\phi \in X^*$ such that $\phi(x) = \inf_{y\in M}\|x - y\|$ and $\|\phi\| = 1$ and $\phi|_M = 0$.
Restrict to the space $M + \mathbb{C}x$ and define $\phi(y + \lambda x) = \lambda\inf_{y\in M}\|x - y\|$. Then $\phi(x) = \inf_{y\in M}\|x - y\|$ and $\phi|_M = 0$.
Since $\phi(x) = \|x\|$, then $1 = \frac{\|x\|}{\|x\|} = \frac{|\phi(x)|}{\|x\|} \leq \|\phi\|$ and

$$|\phi(y + \lambda x)| \leq |\phi(y)| + |\phi(\lambda x)| = 0 + |\lambda||\phi(x)| = |\lambda|\inf_{y\in M}\|x - y\| \leq |\lambda|\|x - \lambda^{-1}y\| = \|\lambda x + y\|.$$

Therefore, $\|\phi\| = \sup_{y+\lambda x}\frac{|\phi(y+\lambda x)|}{\|\lambda x+y\|} \leq 1$ so $\|\phi\| = 1$.
Finally, if we define $p(x) = \|x\|$ for $x \in M + \mathbb{C}x$ then by Hahn-Banach, ϕ can be extended to ψ on all x with $\psi|_{M+\mathbb{C}x} = \phi$. To prove the result, set $M = \{0\}$. □

(b) *Prove that the linear map $\iota : X \to X^{**}$ given by*

$$\iota(x)(\varphi) = \varphi(x) \qquad x \in X, \varphi \in X^*$$

is an isometry.

Proof. Fix $x \in X$, so

$$\|\iota(x)\| = \frac{|\iota(x)(\phi)|}{\|\phi\|_{X^*}} = \sup_{\phi \in X^*} \frac{|\phi(x)|}{\|\phi\|_{X^*}}$$

By part (a), there exists $\phi \in X^*$ such that $\|\phi\|_{X^*} = 1$ and $\phi(x) = \|x\|$, which implies that $\|x\| \leq \|\iota(x)\|_{X^*}$.

Also, for any $\phi \in X^*$, $|\phi(x)| \leq \|\phi\|_{X^*}\|x\|$ and so

$$\|\iota(x)\| \leq \sup_{\phi \in X^*} \frac{|\phi(x)|}{\|\phi\|_{X^*}} \leq \frac{\|\phi\|\|x\|}{\|\phi\|} = \|x\|.$$

So $\|\iota(x)\| = \|x\|$ and so ι is an isometry. □

(c) *A Banach space X is called reflexive if $\iota(X) = X^{**}$. Prove that the Banach space*

$$\ell^1 = \{f \in \mathbb{N} \to \mathbb{C} \mid \|f\|_1 = \sum_k |f(k)| < \infty\}.$$

is not reflexive.

Hint: Consider a weak-$*$ cluster point of the sequence $(\iota(f_n))_{n\in\mathbb{N}} \subseteq (\ell^1)^{**}$, where $f_n \in \ell^2$ is the unit vector

$$f_n(k) = \begin{cases} 1/n & k \leq n \\ 0 & k > n \end{cases}$$

Proof. Compare to Exercise 19 in Chapter 6 of Folland.

We have $\iota(f_n) =: \phi_n \in (\ell^\infty)^*$ is the map

$$\phi_n(f) = n^{-1} \sum_{j=1}^{n} f(j).$$

Note that

$$|\phi_n| \leq n^{-1} \sum_1^n |f(j)| \leq n^{-1} \sum_1^n \|f(j)\|_\infty = \|f(j)\|_\infty,$$

so $\|\phi_n\| \leq 1$ for all $n \in \mathbb{N}$. So (ϕ_n) is in the norm-ball of $(\ell^\infty)^*$, which is wk*-compact by Alaoglu and hence has a cluster point. Define $f_m \in \ell^\infty$ to be

$$f_m(x) := \begin{cases} 0 & x < m \\ 1 & x \geq m \end{cases}.$$

Then $\phi_n(f_m) \to 1$ as $n \to \infty$ for all m. Hence (ϕ_{n_k}) is a convergent subsequence in (ϕ_n) limiting to $\phi \in B((\ell^\infty)^*)$, one must have $\phi(f_m) = 1$ for all m.

Yet if $g \in \ell^1$, then $\sum_1^\infty f_m(j)g(j) = \sum_m^\infty g(j)$ can be made arbitrarily close to 0 for m sufficiently large. So there is no $g \in \ell^1$ such that $\phi(f) = \sum f(j)g(j)$ for all $f \in \ell^\infty$. □

Problem 11. *Let $(g_n)_{n\in\mathbb{N}} \subseteq C([0,1])$ be a sequence of non-negative continuous functions. Assume that for each $k = 0, 1, 2, \ldots$ the limit*

$$\lim_{n\to\infty} \int_0^1 x^k g_n(x)dx \qquad \textit{exists.}$$

Prove that there exists a unique finite positive Radom measure μ on $[0,1]$ such that

$$\int_0^1 f(x)d\mu(x) = \lim_{n\to\infty} \int_0^1 f(x)g_n(x)dx \qquad \textit{for all } f \in C([0,1]).$$

Proof. Define $M := \lim_n \int_0^1 g_n(x)dx < \infty$. Let $\mathcal{A} = \operatorname{span}\{x^k \mid k \in \mathbb{N}\}$. For each $\phi \in \mathcal{A}$, by linearity, $\lim_n \int_0^1 \phi(x)g_n(x)dx$ exists.

By Stone-Weierstrass, $\mathcal{A}$ is dense in $C[0,1]$, so for every $f \in C[0,1]$, $\lim_n \int_0^1 f(x)g_n(x)dx$.

Next, let $\phi : C[0,1] \to \mathbb{C}$ be defined by $\phi(f) = \lim \int_0^1 f(x)g_n(x)dx$. Linearity is obvious. Moreover, for every $f \in C[0,1]$,

$$|\phi(f)| = \left|\lim_n \int_0^1 fg_n(x)dx\right| \le \lim_n \int_0^1 |f(x)||g_n(x)|dx \le \|f\|_n \lim_n \int_0^1 g_n(x)dx = M\|f\|_\infty$$

Hence, ϕ is a bounded linear functional on $C[0,1]$.

By Riesz-Representation, there exists a positive Radon measure μ such that

$$\lim_n \int_0^1 f(x)g_n(x)dx = \phi(f) = \int_0^1 f(x)d\mu(x) \qquad \forall f \in C[0,1].$$

□

Problem 12. *Let X be a locally compact Hausdorff space equipped with a Radon probability measure μ. Let $E \subseteq L^2(X,\mu)$ be a closed linear subspace and assume that E is contained in $C_0(X)$. The goal of this problem is to prove that $\dim(E) < \infty$ by justifying the following steps:*

(a) There exists a constant $1 \le K < \infty$ such that

$$\|f\|_2 \le \|f\|_u \le K\|f\|_2 \quad \textit{for all } f \in E,$$

where $\|\cdot\|_u$ denotes the uniform norm. Hint: us the closed graph theorem for one of the inequalities.

(b) For each $x \in X$, there exists a unique $g_x \in E$ such that $\|g_x\|_2 \le K$ and

$$f(x) = \langle f, g_x \rangle \qquad \text{for all } f \in E.$$

(c) Let $(f_i)_{i\in I}$ be any orthonormal basis for E. Then

$$\sum_{i\in I} |f_i(x)|^2 = \|g_x\|_2^2 \le K^2 \qquad \text{for all } x \in X.$$

(d) $\dim(E) = |I| \le K^2$.

Proof. See January 2017, Problem #5 for a solution to a similar question. □

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Problem 1. *Suppose $U_1, U_2, \ldots$ are open subsets of $[0,1]$. In each case, either prove the statement or disprove it.*

(a) If $\lambda\big(\bigcap_{n=1}^{\infty} U_n\big) = 0$ then for some $n \ge 1$, we have $\lambda(\overline{U_n}) < 1$, where λ is Lebesgue measure and $\overline{U_n}$ is the closure of U_n in the usual topology on $[0,1]$.

Proof. We will disprove this statement. Let $U_n = C_n^c \cap [0,1]$, where $C_n \subset [0,1]$ is a generalized Cantor set of measure $1 - \frac{1}{n}$. Then $m(\bigcap_{n=1}^{\infty} U_n) \le m(U_n) = \frac{1}{n}$, so $m(\bigcap_{n=1}^{\infty} U_n) = 0$. But C_n does not contain an open interval for any n, so $\overline{U_n} = \overline{C_n^c} = [0,1]$ for all n, which has measure 1.

Another counterexample: Let r_m be an enumeration of the rationals on $[0,1]$, and set $a_{n,m} = 1/2^{n+m}$. Set

$$U_n := \bigcup_m (r_m - a_{n,m}, r_m + a_{n,m})$$

These are open since they are a union of open intervals. Moreover, since $\mathbb{Q} \subseteq U_n$ then $\lambda(\overline{U_n}) = \lambda([0,1]) = 1$. But by upper continuity of the Lebesgue measure, then

$$\lambda\left(\bigcap U_n\right) = \lim_m \lambda\left(\bigcup (r_n - a_{n,m}, r_n + a_{n,m})\right) = 0.$$

□

(b) If $\bigcap_{n=1}^{\infty} U_n = \emptyset$, then for some $n \ge 1$, the set $[0,1]\backslash U_n$ contains a non-empty open interval.

Proof. TRUE. Recall that the Baire Category Theorem states that under these assumptions, if each U_n is dense in $[0,1]$ then $\bigcap_{n=1}^{\infty} U_n$ is also dense in $[0,1]$. Then since we have that $\bigcap_{n=1}^{\infty} U_n$ is not dense, then there must be some n such that $[0,1]\backslash U_n$ is not dense. This precisely means that U_n contains a non-empty open interval. □

Problem 2. *Let X be a separable compact metric space and show that $C(X)$ is separable.*

Proof. Remark: If X is a compact metric space, then X is separable. So the separable assumption is superfluous.

Suppose d is the metric on X and (x_n) is a dense countable subset of X. For each $n \in \mathbb{N}$, define the functional f_n by $f_n(x) := d(x, x_n)$. Then each f_n is a continuous functional. Consider $F = \{1, f_1, f_2, \ldots\}$ and consider the subalgebra generated by the rational span of F, call it $\mathbb{Q}[F]$ (this is still countable, we can consider the span, then consider the set where two elements of it are multiplied together, then the set where three elements are multiplied together, etc). This is countable and dense in $\mathcal{A} := \mathbb{R}[F]$. so it is sufficient to show that $\mathcal{A}$ is dense in $C(X)$.

We will attempt to use the Stone Weierstrass Theorem:

By definition, $\mathbb{R}[F]$ contains the constant function 1. We are left to show it separates points. Take two points $x \neq y$ in X. Since $\{x_n\}$ is dense, then there must exist some m such that $d(x, x_m) \leq \frac{1}{3}d(x,y) \neq 0$. If $d(y, x_m) = d(x, x_m)$ then

$$d(x,y) \leq d(x, x_m) + d(y, x_m) = 2d(x, x_m) \leq \frac{2}{3}d(x,y).$$

This cannot be true under our assumption $d(x,y) \neq 0$. So then $f_m(y) = d(y, x_m) \neq d(x, x_m) = f_m(x)$. So f_m separates x and y.

Therefore, by Stone-Weierstrass, $\mathcal{A}$ is dense in $C(X)$. But $\mathbb{Q}[F]$ is countable and dense in $\mathcal{A}$, so therefore $C(X)$ is separable. □

Problem 3. *Let $f : [0,1] \to \mathbb{R}$ be a bounded Lebesgue measurable function such that*

$$\int_0^1 f(t)e^{nt}dt = 0$$

for every $n \in \{0, 1, 2, \ldots\}$. Prove that $f(t) = 0$ for almost every $t \in [0,1]$.

Proof. Using Stone-Weierstrass to show we can pass to the case $\int_0^1 f(t)g(t) = 0$ for all $g \in C[0,1]$ (this convergence is uniform). But $C[0,1]$ is dense in $L^2[0,1]$. So for any $g \in L^2[0,1]$, there is some $(g_n) \subset C[0,1]$ such that $\langle \cdot, g_n \rangle \to \langle \cdot, g \rangle$ in $(L^2)^*$. Hence in fact $\langle f, g \rangle = 0$ for all $g \in L^2[0,1]$, so $f = 0$ a.e.

Another argument: we use Stone-Weierstrass to see $\int_0^1 f(t)g(t) = 0$ for all $g \in C[0,1]$. By a standard density argument, we may pass to the case where g is a step function. We claim that $f = 0$ a.e.

Assume not. WLOG there exists some $E = \{x \in [0,1] \mid f(x) > 0\}$ with $m(E) > 0$ (else consider $-f$).

Since f is bounded, then $E_\infty := \{x \in [1,2] \mid f(x) = \infty\}$ is a null set. Define $E_n := \{x \in [0,1] \mid 1/n < f(x) < n\}$. We can write $E = (\bigcup_n E_n) \cup E_\infty$. So there exists some N such that $m(E_N) = a > 0$.

We can write A as a finite disjoint union of open intervals, $A = \sqcup_{i=1}^m I_i$, such that $m(E_N \triangle A) < \epsilon$ and $A \subseteq E_N$.

Put $g = \sum_{i=1}^{m} \chi_{I_i}$, then $\int_1^2 g(x)f(x) = \int_{E_N} f(x)dx$. Since

$$\left| \int_{E_N} f(x) - \int_A f(x) \right| \leq Nm(E_N \triangle A) < N\epsilon$$

If we choose ϵ small enough, we see the contradiction since $\int_0^1 g(x)f(x) > 0$. □

Problem 4. *(a) Prove that every compact subset of a Hausdorff space is closed.*

Proof. Let A be a compact subset of the Hausdorff space X. To show A is closed, we'll show $A^c = X \backslash A$ is open. Take $x \in X \backslash A$. Then for every $y \in A$, there are disjoint sets U_y and V_y with $x \in V_y$ and $y \in U_y$.

The collection of open sets $\{U_y \mid y \in A\}$ forms an open cover of A. Since A is compact, this open cover has a finite subcover, $U_{y_1}, U_{y_2}, \ldots, U_{y_n}$. Let

$$U := \bigcup_{i=1}^{n} U_{y_i} \qquad V := \bigcap_{i=1}^{n} V_{y_i}$$

Since each U_{y_i} and V_{y_i} are disjoint, then U and V are disjoint. Also, $A \subseteq U$ and $x \in V$.

Thus, for every point $x \in X \backslash A$ we have found an open set V containing x which is disjoint from A. So $X \backslash A$ is open and A is closed. □

(b) Let $f : X \to Y$ be a bijective continuous function between topological spaces. Suppose that X is compact and Y is Hausdorff and prove that f is a homeomorphism.

Proof. Let $g = f^{-1}$. We need to show that g is continuous.

For every $V \subseteq X$, we have $g^{-1}(V) = f(V)$. We want to show that if V is closed in X then $g^{-1}(V)$ is closed in Y.

Suppose V is closed in X. Since X is compact, V is compact by part (a). So $f(V)$ is compact since the continuous image of a compact space is compact.

Since Y is Hausdorff, $f(V)$ is closed by the fact that a compact subspace of Hausdorff space is closed. But $f(V) = g^{-1}(V)$ so $g^{-1}(V)$ is closed. So g is continuous and f is a homeomorphism.

□

(c) Prove or disprove that if X is a dense subset of a topological space Y and if X is Hausdorff in the relative topology, then Y is also Hausdorff.

Proof. FALSE. Consider $Y = \{a, b\}$ with discrete topology $\tau = \{\emptyset, \{a, b\}\}$. Let $X = \{a\}$ with relative topology $\tau_X = \{\emptyset, \{a\}\}$.

Then it's easy to see X is dense in Y (since every open set containing an element of Y has nonempty intersection with X, trivially). Since X has only a single element, it's Hausdorff in the relative topology trivially. But Y is not Hausdorff. □

Problem 5. *Prove that the following limit exists and compute its value:*

$$\lim_{n\to\infty}\int_0^n \left(\sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}\right) e^{-2x}dx.$$

Proof. Let us first note an important simplification of the integrand, by considering the Taylor series expansion of $\cos x$ around a neighbourhood of 0.

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

We now make the following calculation:

$$\int_0^\infty \frac{x^{2k}}{(2k)!} e^{-2x}\, dx = \frac{1}{2}\left(\frac{1}{4}\right)^k.$$

This can be done using integration by parts. Note all terms from this method are of the form $Cx^i e^{-2x}$ for $0 \le i < 2k$ and for some constant C. Since $\lim_{x\to\infty} x^i e^{-2x} = 0$ for all i we have $Cx^i e^{-2x}|_0^\infty = 0 - 0 = 0$ for all $0 < i < 2k$. Hence we only need to find the constant belonging to the e^{-2x} term. Integration by parts gives

$$C = \frac{(2k)!}{2\cdot(-2)^{2k}} = \frac{(2k)!}{2\cdot 4^k},$$

yielding our result. Hence,

$$\sum_{k=0}^{\infty}\int_0^\infty \left|\frac{(-1)^k x^{2k}}{(2k)!} e^{-2x}\right| dx = \sum_{k=0}^{\infty}\int_0^\infty \frac{x^{2k}}{(2k)!} e^{-2x}\, dx = \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k < \infty.$$

We note (by MCT) that $\sum_{i=0}^{\infty} \frac{x^{2k}}{(2k)!} e^{-2x} \in L^1$ is a dominating function for all limiting functions, so

$$\lim_{n\to\infty}\int_0^n \left(\sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}\right) e^{-2x}\, dx \overset{\text{DCT}}{=} \int_0^\infty \lim_{n\to\infty}\left(\sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}\right) e^{-2x}\, dx = \int_0^\infty (\cos x) e^{-2x}.$$

This latter integral is once again solved by integration by parts to yield the value 2/5.

□

Problem 6. *Let X and Y be Banach spaces (over $\mathbb{C}$)*

(a) A linear map $T : X \to Y$ is called adjointable if $T^*f \in X^*$ for every $f \in Y^*$. Prove that T is adjointable if and only if $T \in \mathcal{B}(X,Y)$.

Proof. $\Leftarrow$) if $T \in \mathcal{B}(X,Y)$ then by definition, for every $f \in Y^*$, we have $T^*f \in X^*$

$\Rightarrow$) Suppose $T^*f \in X^*$ for every $f \in Y^*$. We will use the Closed Graph Theorem. Suppose $x_n \to x$ in X and that $Tx_n \to y$ in Y. Then since $T^*f \in X^*$ for every $f \in Y^*$ we can apply this to the convergence to see that

$$f(Tx_n) = (T^*f)(x_n) \to (T^*f)(x) = f(Tx) \qquad \forall f \in Y^*$$

By the Hahn-Banach theorem, Y^* separates points in Y so therefore, $Tx_n \to Tx$. Uniqueness of limits implies $Tx = y$ and so the graph of T is closed. By the Closed Graph Theorem, T is bounded. □

(b) *Suppose a bounded linear functional $\psi : X^* \to \mathbb{C}$ is weak*- continuous. Show (from the definitions) that there exists $x \in X$ such that $\psi(\phi) = \phi(x)$.*

Proof. Define the functional

$$\begin{aligned} \mathrm{ev}_x : X^* &\to \mathbb{C} \\ f &\mapsto f(x) \end{aligned}$$

We want to show that every bounded, linear, weak*-continuous functional $\psi : X^* \to \mathbb{C}$ is of this form.

Indeed, since ψ is weak*-continuous, then it is weak* continuous at 0. Thus, the set $\{f \in X^* \mid |\psi(f)| < 1\}$ is weak* open and must contain a neighborhood of 0. By definition of weak* topology, there must exist $x_1, \dots, x_n \in X$ such that

$$V(x_1, \dots, x_n) := \{f \in X^* \ \mid \ |f(x_i)| \le 1, i = 1, \dots, n\} \subseteq \{f \in X^* \mid |\psi(f)| < 1\}.$$

Then we will next show that $\bigcap_{i=1}^n \ker(\mathrm{ev}_{x_i}) \subseteq \ker(\psi)$.

Indeed, let $f \in \ker(\mathrm{ev}_{x_i})$ so $|f(x_i)| = 0$ for all $i = 1, \dots, n$. Take $\epsilon > 0$ and consider $g = \frac{1}{\epsilon} f$, so $|g(x_i)| = \frac{1}{\epsilon}|f(x_i)| = 0$ for all $i = 1, \dots, n$. In particular, $g \in V(x_1, \dots, x_n)$ and so then we have if $|\psi(g)| < 1$ then $|\psi(f)| < \epsilon$. But ϵ is arbitrary so $\psi(f) = 0$, i.e. $f \in \ker(\psi)$.

Now recall the linear algebra trick that says if for linear functionals $\ker(T) \subseteq \ker(S)$ then S is a scalar multiple of T. In this case, we get that ψ is a linear combination of the ev_{x_i}, i.e. is of the form ev_x where x is a linear combination of the x_i's. If $\psi = \sum_1^n \alpha_i \, \mathrm{ev}_{x_i}$, then $x := \sum_1^n \alpha_i x_i$ is the desired element.

Moreover, because the weak* topology is Hausdorff, x is necessarily unique. □

(c) *Let $S \in \mathcal{B}(Y^*, X^*)$. Prove that S is weak*-weak*-continuous if and only if $S = T^*$ for some $T \in \mathcal{B}(X, Y)$.*

Proof. ⇐) If $S = T^*$ then if $f_\alpha \to f$ is a weak* convergent net in Y^* then for any $y \in Y$, $f_\alpha(y) \to f(y)$. Therefore,

$$\underbrace{(Sf_\alpha - Sf)}_{\in X^*}(x) = (Tx)\underbrace{(f_\alpha - f)}_{\to 0} \to 0.$$

So S is weak*-weak* continuous.

⇒) Suppose $S : Y^* \to X^*$ is weak*-weak* continuous. Then the evaluation function on x, $\mathrm{ev}_x(S)$ is weak* continuous on Y^* (where $\mathrm{ev}_x(S) : Y^* \to \mathbb{C}, (\mathrm{ev}_x(S))(f) = (Sf)(x)$).

By part (b), we know that $\mathrm{ev}_x(S)$ is of the form $\mathrm{ev}_{T(x)}$ for some unique $T(x) \in Y$. Since $T(x)$ is uniquely determined, it follows that T is linear.

We will now check that T is continuous by the closed graph theorem: if $x_n \to x$ adn $Tx_n \to y$ in norm then for each $\phi \in Y^*$ we have

$$\langle \phi, y\rangle = \lim\langle \phi, Tx_n\rangle = \lim\langle S\phi, x_n\rangle = \langle S\phi, x\rangle = \langle \phi, Tx\rangle.$$

And so $y = Tx$ as desired. So T is bounded and therefore, $S = T^*$ is bounded as well. □

Problem 7. *Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions $f_n : [0,1] \to \mathbb{R}$.*

(a) What does it mean for $\{f_n \mid n \geq 1\}$ to be equicontinuous?

Proof. $\{f_n \mid n \geq 1\}$ is said to be equicontinuous if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in [0,1]$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. □

(b) Suppose that for every n, f_n is differentiable and $|f_n'(t)| \leq 1$ for all t. Prove that $\{f_n \mid n \geq 1\}$ is equicontinuous.

Proof. Since $|f_n'(t)| \leq 1$ for all t, then for all n, we have by the mean value theorem that

$$\frac{|f_n(x) - f_n(y)|}{|x - y|} \leq 1.$$

Hence, for any fixed $\epsilon > 0$, setting $\delta = \epsilon$ and for $|x - y| < \delta$ then

$$|f_n(x) - f_n(y)| \leq |x - y| < \delta = \epsilon.$$

□

(c) Suppose the hypothesis of (b) holds and assume in addition that $|f_n(0)| \leq 1$ for every $n \geq 1$. Prove that there exists a continuous function $f : [0,1] \to \mathbb{R}$ and a subsequence $(f_{n(k)})_{k=1}^{\infty}$ converging uniformly to f.

Proof. This is essentially the Arzela-Ascoli Theorem. Since $|f_n(0)| \leq 1$ for all n and since $|f_n'(t)| \leq 1$ for all t, then $|f_n(t)| \leq 2$ for all $t \in [0,1]$ and for all n. That is, $\{f_n\}$ is uniformly bounded. It's also equicontinuous by part (b). Therefore, Arzela-Ascoli theorem states that there is a subsequence $\{f_{n_k}\}$ which converges uniformly. Let f be the limit, and we finish by recalling that the uniform convergence of continuous functions is also continuous.

Note: it might be good to know the Arzela-Ascoli Theorem. □

(d) Show by example that the limit function f need not be differentiable.

Proof. Take $f_n(x) = \sqrt{x^2 + 1/n}$ so $f_n(0) = \frac{1}{\sqrt{n}} \leq 1$ for all n and so $\{f_n\}$ is uniformly bounded. Next, we can see that $f_n'(x) = \frac{x}{\sqrt{x^2+1/n}}$ so that for $x \in [0,1]$ we have $|f_n'(x)| \leq \sqrt{\frac{n}{n+1}} \leq 1$ as desired.

However, it's also clear that the limit must be $f = |x|$ which is not differentiable. □

Problem 8. *Let $\mathcal{H}$ be a complex Hilbert space. Given a non-empty set $E \subseteq \mathcal{H}$ and $x \in \mathcal{H}$, put* $\operatorname{dist}(x, E) = \inf\{\|x - y\| \mid y \in E\}$ *and* $E^{\perp} = \{x \in \mathcal{H} \mid \langle x, y\rangle = 0 \ \forall y \in E\}$.

(a) Let $\mathcal{H}_0 \subseteq \mathcal{H}$ be a closed subspace and $x \in \mathcal{H}$. Prove that there exists $x_0 \in \mathcal{H}_0$ such that $\|x - x_0\| = \operatorname{dist}(x, \mathcal{H}_0)$.

Proof. Let $\delta = \operatorname{dist}(x, \mathcal{H}_0)$. Then there exists a sequence $(y_n) \in \mathcal{H}_0$ such that $\delta_n := \|x - y_n\| \to \delta$. We will show that (y_n) is Cauchy. Indeed,

$$0 \leq \|y_n - y_m\|^2 = -\|y_n + y_m - 2x\|^2 + 2(\|y_n - x\|^2 + \|y_m - x\|^2) \leq -4\delta^2 + 2(\delta_n^2 + \delta_m^2) \to 0.$$

where we use the fact that

$$\|y_n + y_m - 2x\|^2 = 4\left\|\underbrace{\frac{y_n + y_m}{2}}_{\in \mathcal{H}_0} - x\right\|^2 \leq 4\delta.$$

Thus, (y_n) is a Cauchy sequence and so because we are in a Hilbert space, (y_n) converges to some point $x_0 \in \mathcal{H}$. Since $\mathcal{H}_0$ is closed and $y_n \in \mathcal{H}_0$ for all n then we get that $x_0 \in \mathcal{H}_0$. Finally, $\|x - x_0\| = \lim \|x - y_n\| = \lim \delta_n = \delta$.

Exercise: it can be shown if $\mathcal{H}_0$ is convex, then the choice of x_0 is unique! □

(b) With x and x_0 as above, prove that $x - x_0$ is orthogonal to $\mathcal{H}_0$.

Proof. Let $y \in \mathcal{H}_0$ be an arbitrary vector with $\|y\| = 1$, set $\alpha := \langle x - x_0, y\rangle$. Then since $\overline{\alpha}\langle x - x_0, y\rangle = \overline{\alpha}\alpha = |\alpha|^2$ and $\alpha\langle y, x - x_0\rangle = \alpha\overline{\alpha} = |\alpha|^2$, we have

$$\|x - (x_0 + \alpha y)\|^2 = \|x - x_0 - \alpha y\|^2 = \|x - x_0\|^2 - \overline{\alpha}\langle x - x_0, y\rangle - \alpha\langle y, x - x_0\rangle + |\alpha|^2 = \|x - x_0\|^2 - |\alpha|^2.$$

So since $x_0 + \alpha y \in \mathcal{H}_0$ then $\|x - x_0 - \alpha y\| \geq \|x - x_0\|$. Hence $\alpha = 0$.

Therefore, for any nonzero $y \in \mathcal{H}_0$ we can write

$$\langle x - x_0, y\rangle = \|y\|\langle x - x_0, y/\|y\|\rangle = \|y\|0 = 0.$$

So $\langle x - x_0, y\rangle = 0$ for all $y \in \mathcal{H}_0$ so $x - x_0 \perp \mathcal{H}_0$. □

(c) Prove that $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^{\perp}$ (the algebraic direct sum)

Proof. This follows immediately from parts (a) and (b). Take some arbitrary $x \in \mathcal{H}$. We can find the appropriate x_0 as above, so $x = x_0 + (x - x_0) \in \mathcal{H}_0 \oplus \mathcal{H}_0^{\perp}$.

The fact that it is a direct sum follows from the fact that $\mathcal{H}_0 \cap \mathcal{H}_0^{\perp} = \{0\}$. □

(d) Let $E \subseteq \mathcal{H}$ be non-empty. Prove that $(E^{\perp})^{\perp} = E$ if and only if E is a closed subspace.

Proof. If E is closed, then the above parts (a),(b), and (c) apply and prove that $(E^perp)^\perp = E$. To see the converse, we will instead show that $(E^\perp)^\perp = \overline{E}$. The desired result will then immediately follow.

Since $E \subseteq \overline{E}$ then $\overline{E}^\perp \subseteq E^\perp$ and therefore, $(E^\perp)^\perp \subseteq (\overline{E}^\perp)^\perp$. Since $\overline{E}$ is closed, then $(\overline{E}^\perp)^\perp = \overline{E}$ so $(E^\perp)^\perp \subseteq \overline{E}$.

Conversely, since $E^\perp$ is closed for every E (independent of whether E is closed or not) then $(E^\perp)^\perp$ is closed and so since $E \subseteq (E^\perp)^\perp$, then by the monotonicity of topological closure we have that $\overline{E} \subseteq \overline{(E^\perp)^\perp} = (E^\perp)^\perp$.

Therefore, $(E^\perp)^\perp = \overline{E}$. □

Problem 9. *Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. Recall that a Hamel basis for V is a linearly independent subset of V whose linear span equals V.*

(a) Let $S \subseteq V$ and suppose the linear span of S equals V. Show that V has a Hamel basis that is a subset of S.

Proof. Let $\mathcal{C}$ be the collection of linearly independent sets in S. This is non-empty since S is non-empty. A standard Zorn's lemma argument shows that chains have upper bounds in the space, so there is some maximal element B in $\mathcal{C}$. We claim the span of B is V. If there is some v not in the span of B, then since the span of S is V there must be some s not in the span of B either. So $B \cup \{s\}$ is linearly independent, contradicting maximality of B. □

(b) Suppose V has an infinite Hamel basis and show that all hamel bases of V have the same cardinality.

Proof. Suppose that $\{v_i\}_{i\in I}$ and $\{u_j\}_{j\in J}$ are two infinite bases for V. For each $i \in I$, then v_i is in the linear span of $\{u_j\}_{j\in J}$. Therefore, there exists a finite subset $J_i \subseteq J$ such that v_i is in the linear span of the vectors $\{u_j\}_{j\in J_i}$. Therefore, $V = \text{span}(\{v_i\}_{i\in I}) \subseteq \text{span}\{u_j\}_{j\in \bigcup J_i}$. Since no proper subset of $\{u_j\}_{j\in J}$ can span V, it follows that $J = \bigcup_{i\in I} J_i$. Therefore $|J| \le |I|$.

A symmetric argument shows that $|I| \le |J|$. □

Problem 10. *Suppose $(X, \mathcal{M}, \rho)$ is a finite measure space and $\mathcal{A} \subseteq \mathcal{M}$ is an algebra of sets with a finitely additive complex measure $\mu : \mathcal{A} \to \mathbb{C}$ such that $|\mu(E)| \le \rho(E)$ for all $E \in \mathcal{A}$. Show that there exists a complex measure $\nu : \mathcal{M} \to \mathbb{C}$ whose restriction to $\mathcal{A}$ is μ and such that $|\nu(E)| \le \rho(E)$ for all $E \in \mathcal{M}$.*

Hint: you may want to consider the subspace $V \subseteq L^1(\rho)$ that is spanned by the set of characteristic functions χ_E for $E \in \mathcal{A}$, and a certain linear functional on V.

Proof. Solution from Minh Kha.

For each subalgebra $\mathcal{U}$ of $\mathcal{M}$, we define $S_{\mathcal{U}}$ to be the set of all simple functions of the form $\sum_{i=1}^n c_i\chi_{E_i}$ where $c_i \in \mathbb{R}$, $E_i \in \mathcal{U}$. Then $S_{\mathcal{A}}$ is a vector subspace of $S_{\mathcal{M}}$.

Now define $p : S_{\mathcal{M}} \to \mathbb{R}$ such that

$$p(f) = \sup\left\{\sum_{i=1}^n |c_i|\rho(E_i) \;\mid\; f = \sum_{i=1}^n c_i\chi_{E_i}, E_i \cap E_j = \emptyset \;\forall i \ne j, E_i \in \mathcal{M}, c_i \in \mathbb{R}\right\} \quad \forall f \in S_{\mathcal{M}}$$

It's not difficult to check that p satisfies $p(f+g) \le p(f)+p(g)$ for all $f, g \in S_{\mathcal{M}}$ and $p(tf) = tp(f)$ for all $t \in \mathbb{R}^+, f \in S_{\mathcal{M}}$. Thus, p is a seminorm and is just an extension of the total variation of the measure p when you apply to the function $f = 1$.

Define a linear map $T : S_{\mathcal{A}} \to \mathbb{R}$ defined by

$$T(f) = \int_X f d\mu \quad \forall f \in S_{\mathcal{A}}$$

This is linear because of the finite additive property of μ. Then $|T(f)| \le p(f)$ for all $f \in S_{\mathcal{A}}$. By Hahn-Banach, we get a linear extension of T on $S_{\mathcal{M}}$, which we denote by $\widetilde{T}$. Moreover, this extension $\widetilde{T} : S_{\mathcal{M}} \to \mathbb{R}$ satisfies $|\widetilde{T}(f)| \le p(f)$ for all $f \in S_{\mathcal{M}}$.

Now, we define a finite additive measure ν on $\mathcal{M}$ by letting $\nu(E) = \widetilde{T}(\chi_E)$ for all $E \in \mathcal{M}$. Thus, $\nu|_{\mathcal{A}} = \mu$ and $|\nu(E)| \le p(E)$ for all $E \in \mathcal{M}$.

To check the countably additive property of ν, consider any countable collection of disjoint measurable subsets $E_i \in \mathcal{M}$ and so $\chi_{\bigcup_i E_i} = \sum_i \chi_{E_i}$. Thus, $\nu(\bigcup_i E_i) = \sum_i \nu(E_i)$ since the series $\sum_i \widetilde{T}(E_i)$ converges (use $|\widetilde{T}(f)| \le p(f)$ for all $f \in S_{\mathcal{M}}$ and properties of the measure p).

For the complex case, repeat the trick by proving the complex version of the Hahn-Banach theorem from the real version. ☐

14 August 2017

Problem 1. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence of measurable functions on X. Prove, directly from the definition of convergence almost everywhere, that if $\sum_n \mu[|f_n| > 1/n] < \infty$, then the sequence $\{f_n\}$ converges almost everywhere to zero. Deduce that every sequence of measurable functions that converges in measure to zero has a subsequence that converges almost everywhere to zero.*

Proof. Let $E = \{x \in \Omega \mid \lim_n |f_n(x)| = 0\}$. We want $\mu(E^c) = 0$. Let

$$M = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{x \in \Omega \mid |f_n(x)| > 1/n\}$$

Since

$$\mu\left(\bigcup_{n=m}^{\infty} \left\{x \in \Omega \mid |f_n(x)| > \frac{1}{n}\right\}\right) \le \sum_{n=m}^{\infty} \mu\left(\left\{x \in \Omega \mid |f_n(x)| > \frac{1}{n}\right\}\right) \quad \to \quad 0.$$

Therefore, $\mu(M) = 0$ and

$$M^c = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \{x \in \Omega \mid |f_n(x)| \le 1/n\}.$$

Note: $f_n(x) \to 0$ if and only if $\forall \epsilon > 0$, $\exists N$ s.t. $\forall n > N$, $|f_n(x)| < \epsilon$.

So for any $x \in M^c$ choose $1/N < \epsilon$ s.t. $\forall n > N$ we have $|f_n(x)| \leq \frac{1}{n} < \frac{1}{N} < \epsilon$.

Therefore $M^c \subseteq E$, so $E^c \subseteq M$, implying $\mu(E^c) = 0$. So then $\{f_n\}$ converges almost everywhere to zero.

Step 2: We will show that if $f_n \to 0$ in measure, then there exists a subsequence that converges to 0 pointwise almost everywhere.

Suppose for every $\epsilon > 0$, $\mu(\{x \mid |f_n(x)| \geq \epsilon\}) \to 0$. Choose a subsequence $\{f_{n_k}\}$ such that if

$$E_j = \{x \mid |f_{n_j}(x) - f_{n_{j+1}}(x)| > 2^{-j}\}$$

satisfies $\mu(E_j) < 2^{-j}$. Let $F_k = \bigcup_{j=k}^{\infty} E_j$ so $\mu(F_k) \leq \sum_{j=k}^{\infty} 2^{-j} \leq 2^{1-k}$. Let $F = \bigcap_k F_k$ so $\mu(F) = 0$.

For $x \notin F_k$ and for $i \geq j \geq k$ then

$$|f_{n_i}(x) - f_{n_j}(x)| \leq \sum_{\ell=j}^{i-1} |f_{n_\ell}(x) - f_{n_{\ell+1}}(x)| \leq \sum_{\ell=j}^{i-1} 2^{\ell} \leq 2^{-j} \to 0 \quad \text{as } k \to \infty.$$

So f_{n_k} is pointwise Cauchy on $x \notin F$, so let

$$f(x) = \begin{cases} \lim f_{n_k}(x) & x \notin F \\ 0 & \text{otherwise} \end{cases}$$

So $f_{n_k} \to 0$ almost everywhere and $f_n \to f$ in measure since

$$\mu(\{x \mid |f_n(x) - f(x)| \geq \epsilon\}) \leq \underbrace{\mu(\{x \mid |f_n(x) - f_{n_\ell}(x)| \geq \epsilon/2\})}_{\to 0} + \underbrace{\mu(\{x \mid |f_{n_\ell}(x) - f(x)| \geq \epsilon\})}_{\to 0}$$

and

$$\mu(\{x \mid |f(x)| \geq \epsilon\}) \leq \underbrace{\mu(\{x \mid |f(x) - f_n(x)| \geq \epsilon/2\}}_{\to 0} + \underbrace{\mu(\{x \mid |f_n(x)| \geq \epsilon/2\})}_{\to 0}$$

so $f = 0$ almost everywhere. Thus, $\{f_{n_k}\}$ converges to 0 almost everywhere. □

Problem 2. *Show that there is a sequence of nonnegative functions $\{f_n\}$ in $L^1(\mathbb{R})$ such that $\|f_n\|_{L^1(\mathbb{R})} \to 0$, but for any $x \in \mathbb{R}$, $\limsup_n f_n(x) = \infty$.*

Proof. We will explicitely construct such a sequence. Consider the following pattern:

To cover $[-1, 1]$ let

$$f_1 = \sqrt{1}\chi_{[-1,0]}, \qquad f_2 = \sqrt{1}\chi_{[0,1]}.$$

so that $\|f_1\|_{L^1(\mathbb{R})} = 1 = \|f_2\|_{L^1(\mathbb{R})}$. To cover $[-2,2]$, next let

$$f_3 = \sqrt{2}\chi_{[-2,-1.5]}, \qquad f_4 = \sqrt{2}\chi_{[-1.5-1]} \qquad \ldots, f_{10} = \sqrt{2}\chi_{[1.5,2]}$$

so then $\frac{1}{\sqrt{2}} = \|f_3\|_{L^1(\mathbb{R})} = \|f_4\|_{L^1(\mathbb{R})} = \ldots = \|f_10\|_{L^1(\mathbb{R})}$. Next, we cover $[-3,3]$ so that

$$f_{11} = \sqrt{3}\chi_{[-3,-2.666]}, \ldots f_{28} = \sqrt{3}\chi_{[2.666,3]}$$

so that $\frac{1}{\sqrt{3}} = \|f_{11}\|_{L^1(\mathbb{R})} = \ldots = \|f_{28}\|_{L^1(\mathbb{R})}$. If we continue in this fashion, we get the desired functions.

Explicitely, for $n = \sum_{i=1}^{N-1} 2i^2 + k = \frac{1}{3}(N-1)(N)(2N-1) + k$ where $N \in \mathbb{N}$, $0 \leq k < 2N^2$, then we set

$$f_n = \sqrt{N}\chi_{[-N+k/N, -N+(k+1)/N]}$$

so that $\|f_n\|_{L^1(\mathbb{R})} = \frac{1}{\sqrt{N}}$ but for every $x \in \mathbb{R}$, it's clear that $\limsup_n f_n(x) = \infty$. □

Problem 3. *Construct a sequence of nonnegative Lebesgue measurable functions $\{f_n\}$ on $[0,1]$ such that*

(a) $f_n \to 0$ almost everywhere, and

(b) for any interval $[a,b] \subseteq [0,1]$,

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = b - a.$$

Proof. We will prove a result in greater generality: there exists a sequence (f_n) that converges to 0 a.e. such that $\int f_n f \to \int f$. After proving this statement we will explain how to adapt this argument to take a shorter amount of time for the qual. Before beginning, we recall "little-o" notation: we say a sequence $a_n = o(f(n))$ if $\lim_n \frac{a_n}{f(n)} = 0$.

Claim. For any $f \geq 0$ that is continuous on $[0,1]$, $|n \int_{[x,x+\frac{1}{n^2}]}(f(y) - f(x))dy| = o(\frac{1}{n})$.

For any $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $0 < \delta < \frac{1}{N}$ then $|f(x+\delta) - f(x)| < \varepsilon$ for all $x \in [0,1]$. Hence for $n > N$, $n\int_{[x,x+\frac{1}{n^2}]} |f(y) - f(x)|\, dy < n\epsilon\frac{1}{n^2} = o(\frac{1}{n})$, proving the claim.

By the exact same method as taken in the claim, $|\int_{[x,x+\frac{1}{n}]}(f(y) - f(x))\, dy| = o(\frac{1}{n})$ as well. We will use both in what is to come.

Let $f_n(x) := \sum_{k=0}^{n-1} n\chi_{[\frac{k}{n}, \frac{k}{n}+\frac{1}{n^2}]}$, and let ε, N be as above. Clearly f_n is measurable and satisfies (a) above. To prove (b), observe that, for $n > N$,

$$\begin{aligned} I_k &:= \left| \int_{\frac{k}{n}}^{\frac{k+1}{n}} \left[n f \chi_{[\frac{k}{n}, \frac{k}{n}+\frac{1}{n^2}]} - f \right] \right| \\ &= \left| n \int_{\frac{k}{n}}^{\frac{k}{n}+\frac{1}{n^2}} f - \int_{\frac{k}{n}}^{\frac{k+1}{n}} f \right| \\ &= \left| n \int_{\frac{k}{n}}^{\frac{k}{n}+\frac{1}{n^2}} f(y) - f\left(\frac{k}{n}\right) dy - \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(y) - f\left(\frac{k}{n}\right) dy \right| \\ &= o\left(\frac{1}{n}\right). \end{aligned}$$

Hence $|\int (f f_n - f)| \leq \sum_{k=0}^{n-1} I_k = o(1)$.

This holds for all $f \geq 0$ which is continuous, so in particular, it will hold for $f = \chi_{[a,b]}$.

Shorter proof: Taking the same sequence and the same I_k as above, note that for each n, there are at most two values of k for which I_k is non-zero (namely, those whose intervals of integration which intersect with the endpoints a and b). It is easy to see that $|I_k| \leq \frac{2}{n}$ for all k by its definition and a quick triangle inequality, and this completes the proof. Note we did not use the claim, the statement following it, or the lengthy calculation following the definition of I_k for this argument. (We needed slightly stricter bounds for I_k in the continuous case, which is where most of the work above comes from.) □

Problem 4. *In this problem the measure is Lebesgue measure on* $[0,1]$. *The norm on* $L^\infty[0,1]$ *is the essential supremum norm, which for a continuous function is the same as the supremum norm.*

(a) Prove or disprove that $L^\infty[0,1]$ *is separable in the norm topology.*

Proof. $L^\infty[0,1]$ is not separable in the norm topology. Consider the collection of functions $f_r = \chi_{[-r,r]}$ for real $1 \geq r > 0$. Since there are uncountably many such r and since $\|f_r - f_{r'}\|_\infty = 1$ for any $r \neq r'$, it's impossible to have a countable subset of $L^\infty[0,1]$ that is dense in it. □

(b) Recall that $L^\infty[0,1] = (L^1[0,1])^*$. *What is the weak* closure in* $L^\infty[0,1]$ *of the unit ball of* $C[0,1]$*? Prove your assertion.*

Proof. We anticipate that the largest this weak* closure could be is the unit ball of $L^\infty[0,1]$; we will prove this is the case.

First, since $B(C[0,1]) \subset B(L^\infty[0,1])$,

$$\overline{B(C[0,1])}^{w^*} \subset \overline{B(L^\infty[0,1])}^{w^*} \overset{\text{Alaoglu}}{=} B(L^\infty[0,1]).$$

<u>Claim:</u> $B(C[0,1])$ is weak*-dense in $B(L^\infty[0,1])$.

This is guaranteed by Lusin's theorem. Let $f \in L^\infty[0,1]$ with $\|f\|_\infty \leq 1$. Let $(f_n) \subset C[0,1]$ be such that $\|f_n\|_\infty \leq \|f\|_\infty$ and $f_n = f$ except on a set A_n such that $\mu(A_n) < \frac{1}{n}$. For $g \in L^1$

and $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that $\int |g| < \varepsilon$ whenever $\mu(N) < \frac{1}{N}$ (see Corollary 3.6 of Folland). So

$$\int |f - f_N||g| = \int_{A_N^c} |f - f_N||g| + \int_{A_N} |f - f_N||g| < 2\|f\|_\infty \varepsilon.$$

This means $B(L^\infty[0,1])$ is contained in the weak*-closure of $B(C[0,1])$, completing the claim.

□

Problem 5. *Prove that if $a_1, a_2, \ldots, a_N$ are complex numbers, then*

(a) $\int_0^1 |\sum_{k=1}^N a_k \exp(2\pi ikt)|^p dt \leq \sum_{k=1}^N |a_k|^p$, *if* $1 \leq p \leq 2$, *and*

(b) $\int_0^1 |\sum_{k=1}^N a_k \exp(2\pi ikt)|^p dt \geq \sum_{k=1}^N |a_k|^p$, *if* $2 \leq p < \infty$.

Proof. Note first the following facts:

- $\{\exp(2\pi ikt)\}$ is orthonormal in L^2
- For a finite measure space and $p \leq q$, then

$$\|f\|_p \leq \mu(X)^{1/p-1/q}\|f\|_q$$

- For a discrete X and $p \leq q$, $\|f\|_q \leq \|f\|_p$.

Since $\{\exp(w\pi ikt)\}$ is orthonormal, then

$$\left\|\sum_{k=1}^N a_k \exp(2\pi ikt)\right\|_2^2 = \sum_{k=1}^N |a_k|^2.$$

Then if we let $a = (a_1, \ldots, a_N)$ and $f = \sum_{k=1}^N a_k \exp(2\pi ikt)$, we see that for $1 \leq p \leq 2$, we have

$$\int_0^1 \left|\sum_{k=1}^N a_k \exp(2\pi ikt)\right|^p dt = \|f\|_p^p \leq \|f\|_2^p = \|a\|_2^p \leq \|a\|_p^p.$$

To see (b), then similarly for $2 \leq p < \infty$,

$$\int_)^1 \left|\sum_{k=1}^N a_k \exp(2\pi ikt)\right|^p dt = \|f\|_p^p \geq \|f\|_2^p = \|a\|_2^p \geq \|a\|_p^p.$$

□

Problem 6. *Prove that if X is an infinite dimensional Banach space and X^* is separable in the norm topology, then there is a sequence $\{x_n\}$ of norm one vectors in X such that $\{x_n\}$ converges weakly to zero.*

Proof. Suppose $\{x_n^*\}$ is a dense, countable subset of X^*.

Claim: For every n, then $\bigcap_{k=1}^n \ker(x_k^*)$ is non-trivial.

Indeed, assume to the contrary that $\bigcap_{k=1}^n \ker(x_k^*) = \{0\}$. Then the map

$$\begin{aligned} F : X &\to \mathbb{F}^n \\ x &\mapsto (x_1^*(x), \dots, x_n^*(x)) \end{aligned}$$

is linear and injective. Let $\{e_1, \dots, e_m\}$ be a basis for $F(X)$. Choose $y_k \in F^{-1}(\{e_k\})$. For all $x \in X$, we can write $F(x) = \sum_{i=1}^m a_i e_i$. so $F(x - \sum a_i y_i) = \sum a_i e_i - \sum a_i e_i = 0$ so x is in the span and then X must be finite dimensional, contradiction! So the claim holds.

Now, choose $x_n \in S_X \cap (\bigcap_{k=1}^n \ker(x_k))$. Fix $x^* \in X^*$, $\epsilon > 0$, so $\exists N \in \mathbb{N}$ such that $\|x^* - x_N^*\| < \epsilon$. Then for all $n \geq N$, $x_n \in \ker(x_N^*)$ so

$$|x^*(x_n)| = |(x^* - x_n^*)(x_n)| \leq \|x^* - x_N^*\| < \epsilon$$

So then $x^*(x_n) \to 0$. □

Problem 7. *Prove or disprove each of the following statements.*

(a) If $\{f_n\}$ is a sequence in $C[0,1]$ that converges weakly, then also $\{f_n^2\}$ converges weakly.

Proof. YES. Recall that $f_n \in C[0,1]$ converges weakly if and only if it converges pointwise and is uniformly bounded.

Suppose $f_n \to f$ weakly, let $M := \sup_n \|f_n\| < \infty$. Then $f_n \to f$ pointwise so $f_n^2 \to f^2$ pointwise and $\sup_n \|f_n^2\| = M^2 < \infty$. So $f_n^2 \to f^2$ weakly. □

(b) If $\{f_n\}$ is a sequence in $L^2[0,1]$ that converges weakly, then also $\{f_n^2\}$ converges weakly. (Lebesgue measure on $[0,1]$)

Proof. NO. Take $f_n(x) = x^{-1/3}\chi_{[1/n,1]}(x)$, so $f_n \to f = x^{-1/3}$ in norm but $f_n^2(x) = x^{-2/3}\chi_{[1/n,1]}(x)$ but

$$\int_0^1 f_n^2(x) x^{-1/3} dx = \int_0^1 x^{-1}\chi_{[1/n,1]} = \log(n) \to \infty.$$

□

Problem 8. *Let $\{f_n\}$ be a sequence of continuous functions on $\mathbb{R}$ that converges pointwise to a real valued function f. Prove that for each $a < b$, the function f is continuous at some point of $[a,b]$.*

Hint: Let $E_{n,m,k} = [|f_n - f_m| \leq 1/k]$.

Proof. Fix some $[a,b] \subseteq [0,1]$. By Egoroff's Theorem, $f_n \to f$ uniformly outside a set of measure $\frac{b-a}{2}$. Then f must be continuous outside of this set.

Note: Likely, the question was meant to prove Egoroff's theorem, see Folland for that proof! □

Problem 9. *Let X and Y be compact Hausdorff spaces and let S be the set of all real functions on $X \times Y$ of the form $h(x,y) = f(x)g(y)$ with f in $C(X)$ and g in $C(Y)$.*

Prove or disprove that the linear span of S is dense in $C(X \times Y)$.

Proof. We will use Stone-Weirstrass theorem here. Note that if $h_1(x,y) = f_1(x)g_1(y)$ and $h_2(x,y) = f_2(x)g_2(y)$ are two functions in S, then

$$(h_1h_2)(x,y) = h_1(x,y)h_2(x,y) = f_1(x)g_1(y)f_2(x)g_2(y) = (f_1f_2)(x)(g_1g_2)(y)$$

where if $f_1, f_2 \in C(X)$ then so is f_1f_2 (and similarly, $g_1g_2 \in C(Y)$). So then S is an algebra. Thus, it follows that $\text{span}(S)$ is an algebra as well.

Next, S separates points. Indeed, suppose $(x,y) \neq (x',y')$ in $X \times Y$. If $x \neq x'$ then choose some $f \in C(X)$ that separates x and x'. Take $g \in C(Y)$ to be the constant function $g = 1$. Then letting $h(x,y) := f(x)g(y) = f(x)$, h separates the two points. If $x = x'$ then $y \neq y'$ so the same trick works, setting $f = 1 \in C(X)$ and choosing g to separate y and y', letting $h(x,y) := f(x)g(y) = g(y)$ to then separate points.

Therefore, by the Stone-Weierstass theorem, $\text{span}(S)$ is dense in $C(X \times Y)$. □

Problem 10. *Let X be a Hilbert space and assume that $\{x_n\}$ is a sequence in X that converges weakly to zero. Prove that there is a subsequence $\{y_k\}$ of $\{x_n\}$ such that the sequence $\|N^{-1}\sum_{k=1}^{N} y_k\|$ converges to zero.*

Caution: the same statement is NOT true in all Banach spaces, not even in all reflexive Banach spaces.

Proof. Note: This is the Banach-Saks Theorem

We shall successively choose the n_k in the following manner. Beginning for definiteness with $n_1 = 1$, let n_2 be the first index such that $|\langle f_1, f_n\rangle| \leq 1$ (this choice is possible since $\langle f_1, f_n\rangle \to 0$ as $n \to \infty$). In general, after having chosen $f_{n_1}, f_{n_2}, \ldots, f_{n_k}$, we choose n_{k+1} so that

$$|\langle f_{n_1}, f_{n_{k+1}}\rangle| \leq \frac{1}{k}, \ldots, |\langle f_{n_k}, f_{n_{k+1}}\rangle| \leq \frac{1}{k}$$

Since $\{f_n\}$ converges weakly, then it is bounded and so $\|f_n\|$ forms a bounded sequence, say $\|f_n\| \leq M$ so by expanding the inner product, we get

$$\left\|\frac{f_{n_1} + f_{n_2} + \ldots + f_{n_k}}{k}\right\|^2 \leq \frac{kM^2 + 2\times 1 + 4 \times \frac{1}{2} + \ldots + 2(k-1)\times\frac{1}{k-1}}{k^2} < \frac{M^2+2}{k}$$

which then implies

$$\left\| \frac{f_{n_1} + f_{n_2} + \ldots + f_{n_k}}{k} \right\|^2 \to 0.$$

□

Problem 11. *Let $F \subseteq C([0,1])$ be a family of continuous functions such that*

(a) the derivative $f'(t)$ exists for all $t \in (0,1)$ and $f \in F$.

(b) $\sup_{f\in F} |f(0)| < \infty$ and $\sup_{f\in F} \sup_{t\in(0,1)} |f'(t)| < \infty$.

Prove that F is precompact in the Banach space $C([0,1])$ equipped with the norm $\|f\| = \sup_{t\in[0,1]} |f(t)|$.

Proof. We will use the Arzela-Ascoli Theorem.

To see F is equicontinuous, fix some $\epsilon > 0$, and let $\delta = \frac{\epsilon}{M}$ where $M = \sup_{f\in F} \sup_{t\in(0,1)} |f'(t)| < \infty$. Then by the mean value theorem, for any $a < b$, there exists some $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$ so that $|f(b) - f(a)| \le |f'(c)||b-a| \le M|b-a| < M\delta = \epsilon$.

To see F is pointwise bounded, we see that for any $b \in [0,1]$, then for some $c \in [0,b]$, we have $f(b) = f'(c)b + f(0)$, so that

$$|f(b)| \le M + \sup_{f\in F} |f(0)|.$$

That is, F is uniformly bounded!

Then by Arzela-Ascoli, $\overline{F}$ is compact. □

Problem 12. *Let $\{x_n\}$ be a weakly Cauchy sequence in a normed linear space X. Prove that*

(a) x_n is norm bounded in X

Proof. Let c denote the space of convergent sequences, and consider the map

$$\begin{aligned} T : X^* &\to c \\ x^* &\mapsto (x^*(x_n)) \end{aligned}$$

Note $\widehat{x_n}(x^*) = x^*(x_n)$ is convergent for all $x^* \in X^*$, so $\sup_n |\widehat{x_n}(x^*)| < \infty$ for all x^*. By Uniform Boundedness, $\sup_n \|\widehat{x_n}\| = \sup_n \|x_n\| < \infty$. □

(b) There exists x^{**} in X^{**} such that x_n converges weak to x^{**}, and $\|x^{**}\| \le \liminf \|x_n\|$.*

Proof. Since (x_n) is weakly Cauchy, then for every $x^* \in X^*$ the sequence $(x^*(x_n))$ is Cauchy, hence convergent. We can define

$$\begin{aligned} x^{**} : X^{**} &\to \mathbb{C} \\ x^* &\mapsto \lim_n x^*(x_n) \end{aligned}$$

Uniform boundedness shows that $\|x_n\|$ is bounded, hence x^{**} is bounded. Finally,

$$|x^{**}(x^*)| = \liminf |x^*(x_n)| \leq \liminf \|x^*\|\|x_n\| = \Big(\liminf \|x_n\|\Big)\Big(\|x^*\|\Big).$$

So then $\|x^{**}\| \leq \liminf \|x_n\|$. □

15 January 2017

Problem 1. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Prove directly from the definition of convergence almost everywhere that if for all n, $\mu\left(\left\{x \in \Omega \mid |f_n(x)| > \frac{1}{n}\right\}\right) < n^{-3/2}$, then $f_n \to 0$ μ-a.e.*

Proof. Let $E = \{x \in \Omega \mid \lim_n |f_n(x)| = 0\}$. We want $\mu(E^c) = 0$. Let

$$M = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{x \in \Omega \mid |f_n(x)| > 1/n\}$$

Since

$$\mu\left(\bigcup_{n=m}^{\infty} \{x \in \Omega \mid |f_n(x)| > \frac{1}{n}\}\right) \leq \sum_{n=m}^{\infty} \mu\left(\left\{x \in \Omega \mid |f_n(x)| > \frac{1}{n}\right\}\right) < \sum_{n=m}^{\infty} n^{-3/2} \to 0.$$

Therefore, $\mu(M) = 0$ and

$$M^c = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \{x \in \Omega \mid |f_n(x)| \leq 1/n\}.$$

Note: $f_n(x) \to 0$ if and only if $\forall \epsilon > 0$, $\exists N$ s.t. $\forall n > N$, $|f_n(x)| < \epsilon$.

So for any $x \in M^c$ choose $1/N < \epsilon$ s.t. $\forall n > N$ we have $|f_n(x)| \leq \frac{1}{n} < \frac{1}{N} < \epsilon$.

Therefore $M^c \subseteq E$, so $E^c \subseteq M$, implying $\mu(E^c) = 0$. □

Problem 2. *Find all f in $L^1(1,2)$ such that for every natural number n we have $\int_1^2 x^{2n} f(x)dx = 0$. Give reasons for all assertions you make.*

Proof. Let $f(x) = 0$ on $x = 1, 2$. We now consider $f \in L^1[1,2]$. Using Stone-Weierstrass to show we can pass to the case $\int_1^2 g(x)f(x) = 0$ for all $g \in C[1,2]$.

By a standard density argument, we may pass to the case where g is a step function. We claim that $f = 0$ a.e.

Assume not. WLOG there exists some $E = \{x \in [1,2] \mid f(x) > 0\}$ with $m(E) > 0$ (else consider $-f$).

Since $f \in L^1[1,2]$ then $E_\infty := \{x \in [1,2] \mid f(x) = \infty\}$ is a null set. Define $E_n := \{x \in [1,2] \mid 1/n < f(x) < n\}$. We can write $E = (\bigcup_n E_n) \cup E_\infty$. So there exists some N such that $m(E_N) = a > 0$.

We can write A as a finite disjoint union of open intervals, $A = \sqcup_{i=1}^m I_i$, such that $m(E_N \triangle A) < \epsilon$ and $A \subseteq E_N$.

Put $g = \sum_{i=1}^m \chi_{I_i}$, then $\int_1^2 g(x)f(x) = \int_{E_N} f(x)dx$. Since

$$\left| \int_{E_N} f(x) - \int_A f(x) \right| \leq N m(E_N \triangle A) < N\epsilon$$

If we choose ϵ small enough, we see the contradiction since $\int_1^2 g(x)f(x) > 0$. □

Problem 3. *A. Prove that there exists a sequence of measurable functions g_n on $[0,1]$ such that*

(a) $g_n(x) \geq 0$ *for any* $x \in [0,1]$*;*

(b) $\lim_n g_n(x) = 0$ *a.e.;*

(c) For any continuous function $f \in C[0,1]$*,*

$$\lim_{n\to\infty} \int_0^1 f(x)g_n(x)dx = \int_0^1 f(x)dx.$$

Proof. (Solution from Ting Lu, TeX-ed by John Weeks)

It suffices to assume f is non-negative. Any $f \in C[0,1]$ is uniformly continuous since $[0,1]$ is compact. The following lemma will then come in handy:

Claim. With f as above, $n\int_{[x,x+\frac{1}{n^2}]}(f(y) - f(x))dy = o(\frac{1}{n})$.

For any $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $0 < x < \frac{1}{N}$ then $f(x) - f(0) < \varepsilon$. Hence for $n > N$, $n\int_{[0,\frac{1}{n^2}]}(f(x) - f(0)) < n\epsilon\frac{1}{n^2}$.

A clear extension to this is that $\int_{[x,x+\frac{1}{n}]}(f(y) - f(x))dy = o(\frac{1}{n})$.

Let $g_n(x) := \sum_{k=0}^{n-1} n\chi_{[\frac{k}{n},\frac{k}{n}+\frac{1}{n^2}]}$, and let ε, N be as above. Clearly g_n is measurable and satisfies (a) and (b) above. To prove (c), observe that, for $n > N$,

$$\begin{aligned} I_k &:= \left| \int_{\frac{k}{n}}^{\frac{k+1}{n}} nf\chi_{[\frac{k}{n},\frac{k}{n}+\frac{1}{n^2}]} - f \right| \\ &= \left| n\int_{\frac{k}{n}}^{\frac{k}{n}+\frac{1}{n^2}} f - \int_{\frac{k}{n}}^{\frac{k+1}{n}} f \right| \\ &= o\left(\frac{1}{n}\right). \end{aligned}$$

Hence $|\int(fg_n - f)| \leq \sum_{k=0}^{n-1} I_k = o(1)$. □

B. *If g_n is a sequence of measurable functions on $[0,1]$ such that (a), (b), and (c) are satisfied, what can you say about $\int_0^1 \sup_n g_n(x)dx$?*

Proof. TBD □

Problem 4. *We say that a sequence $\{a_n\}_{n=1}^{\infty}$ in $[0,1]$ is equidistributed (in $[0,1]$) if and only if for all intervals $[c,d] \subset [0,1]$,*

$$\lim_{n\to\infty} \frac{|\{a_1,\dots,a_n\} \cap [c,d]|}{n} = d - c.$$

(Here $|A|$ denotes the number of elements in the set A.)

Let $\mu_N = \frac{1}{N}\sum_{1\leq n\leq N} \delta_{a_n}$ with δ_{a_n} the point measure at a_n, that is, for any subset $S \in [0,1]$, $\delta_{a_n}(S) = \begin{cases} 1 & \text{if } a_n \in S \\ 0 & \text{if } a_n \notin S \end{cases}$.

Show that $\{a_n\} \subset [0,1]$ is equidistributed if and only if

$$\lim_{N\to\infty} \int_0^1 f d\mu_N = \int_0^1 f dm,$$

for all continuous functions on $[0,1]$, where m is Lebesgue measure.

Proof. Note that $\{a_n\}$ is equidistributed if and only if

$$\lim_n \frac{|\{a_1,\dots,a_n\} \cap [c,d]|}{n} = d - c$$

if and only if

$$\lim_n \int_0^1 f d\mu_N = \int_0^1 f dm \quad \text{for } f \text{ simple functions (since we can take } f = \chi_{[c,d]})$$

$\Rightarrow$) It's easy to see if $\{a_n\}$ is equidistributed for $f = \chi_{[c,d]}$.

$$\lim_N \int_0^1 f d\mu_N = \lim_N \frac{|\{a_1,\dots,a_N\} \cap [c,d]|}{N} = d - c = \int_0^1 f dm$$

Thus, "=" holds for step functions.

Using Darboux's definition of integral for $f \in C[0,1]$, $\forall \epsilon > 0$ there exists step functions f_1, f_2 such that $f_1 \leq f \leq f_2$ and $\int_0^1 (f_2 - f_1)dx < \epsilon$ where the lower sum is

$$\int_0^1 f_1(x)dx = \lim_N \frac{1}{N}\sum_1^N f_1(a_n) \le \liminf_N \frac{1}{N}\sum_1^N f(a_n)$$

and the upper sum is

$$\int_0^1 f_2(x)dx = \lim_N \frac{1}{N}\sum_1^N f_2(a_n) \ge \limsup_N \frac{1}{N}\sum_1^N f(a_n)$$

Then

$$\left|\limsup_N \frac{1}{N}\sum_1^N f(a_n) - \liminf_N \frac{1}{N}\sum_1^N f(a_n)\right| \le \epsilon.$$

Therefore $\lim_N \frac{1}{N}\sum_1^N f(a_n)$ exists and by definition must be $\int_0^1 f d\mu$.

$\Leftarrow$) If we know $\lim_K \int_0^1 g_n d\mu_K = \int_0^1 g_n d\mu$ for all $g_n \in C[0,1]$. Let $f = \chi_{[c,d]}$, choose $g_n \to f$ in L^1 and each $g_n \searrow f$ positive, $g_n \in C[0,1]$ with $g_n|_{[c,d]} = 1 = f|_{[c,d]}$.

We want to show $\lim_K \int_0^1 f d\mu_K = \int_0^1 f dm$. Indeed,

$$\begin{aligned}
\left|\int_0^1 f d\mu_K - \int_0^1 f dm\right| &= \left|\int_c^d f d\mu_K - \int_c^d f dm\right| \\
&= \left|\int_c^d g_n d\mu_K - \int_c^d f dm\right| \\
&= \int_c^d g_n d\mu_K - \int_c^d f dm \\
&\le \int_0^1 g_n d\mu_K - \int_0^1 f dm \\
&\le \left|\int_0^1 g_n d\mu_K - \int_0^1 g_n dm\right| + \left|\int_0^1 g_n dm - \int_0^1 f dm\right| \to 0.
\end{aligned}$$

□

Problem 5. *Consider the space $C([0,1])$ of real-valued continuous functions on the unit interval $[0,1]$. We denote by $\|f\|_\infty := \sup_{x\in[0,1]} |f(x)|$ the supremum norm of $f \in C([0,1])$ and by $\|f\|_2 := (\int_0^1 |f(x)|^2 dx)^{\frac{1}{2}}$ the L^2-norm of $f \in C([0,1])$. Let S be a closed linear subspace of $(C([0,1]), \|\cdot\|_\infty)$. Show that if S is complete in the norm $\|\cdot\|_2$, then S is finite-dimensional.*

Proof. Let $T : (S, \|\cdot\|_2) \to (S, \|\cdot\|_\infty)$ by $T(x) = x$. Note that both spaces are complete.

Assume $x_n \to x$ in $\|\cdot\|_2$ and $T(x_n) \to y$ in $\|\cdot\|_\infty$ then

$$\|T(x_n) - y\|_2 \leq \|T(x_n) - y\|_\infty \to 0.$$

So

$$\|x - y\|_2 \leq \|x - T(x_n)\|_2 + \|T(x_n) - y\|_2 \leq \|x - x_n\|_2 + \|T(x_n) - y\|_\infty \to 0$$

so $x = T(x) = y$.

Therefore, by closed graph theorem, we know T is bounded. So there exists some C such that $\|f\|_\infty \leq C\|f\|_2$.

Now let $f_1, \ldots, f_n$ be an orthonormal family in S. Then for all fixed $x \in [0,1]$

$$f_1(x)^2 + \cdots + f_n(x)^2 \leq \|f_1(x)f_1 + \cdots + f_n(x)f_n\|_\infty \leq C\|f_1(x)f_1 + \cdots + f_n(x)f_n\|_2$$

So then because f_n's are orthogonal and $\|f_k\|_2^2 = 1$,

$$(f_1(x)^2 + \cdots + f_n(x)^2)^2 \leq C^2 \left(f_1(x)^2\|f_1\|_2^2 + \cdots + f_n(x)^2\|f_n\|_2^2\right) = C^2(f_1(x)^2 + \cdots + f_n(x)^2)$$

Then $f_1(x)^2 + \cdots + f_n(x)^2 \leq C^2$. So

$$n = \int_0^1 f_1(x)^2 + \cdots + f_n(x)^2 dx \leq \int_0^1 C^2 dx = C^2 \Rightarrow n \leq C^2$$

Thus, the number of orthogonal family in S is at most C^2. So S is finite dimensional. □

Problem 6. *Prove that if a function $f : [0,1] \to \mathbb{R}$ is Lipschitz, with*

$$|f(x) - f(y)| \leq M|x - y|$$

for all $x, y \in [0,1]$, then there is a sequence of continuously differentialbe functions $f_n : [0,1] \to \mathbb{R}$ such that

(i) $|f_n'(x)| \leq M$ *for all* $x \in [0,1]$*;*

(ii) $f_n(x) \to f(x)$ *for all* $x \in [0,1]$*.*

Proof. It's easy to prove f is absolutely continuous $\Rightarrow$ f is of bounded variable $\Rightarrow$ f is differentiable a.e. $\Rightarrow$ f' exists a.e.

Also, when f' exists, $|f'(x)| \leq M$.

Then there exists simple $\phi_1, \phi_2, \ldots$ such that $0 \leq |\phi_1| \leq |\phi_2| \leq \cdots \leq |\phi_n| \leq \cdots \leq |f'| \leq M$ and $\phi_n \to f'$ uniformly on $[0,1]$ where f' exists. Define

$$f_n(x) := \int_0^x \phi_n(t)dt + f(0) \qquad f(x) := \int_0^x f'(t)dt + f(0)$$

Then $|f_n'(x)| = |\phi_n(x)| \leq M$ and for all $x \in [0,1]$,

$$|f_n(x) - f(x)| \leq \int_0^x |\phi_n(t) - f'(t)|dt \to 0$$

since ϕ_n converges to f' uniformly. □

Problem 7. *Given $f : \mathbb{R} \to \mathbb{R}$ bounded and uniformly continuous and K_n with $K_n \in L^1(\mathbb{R})$ for $n = 1, 2, 3, \ldots$ such that*

(i) $\|K_n\|_1 \leq M < \infty$, $n = 1, 2, 3, \ldots$

(ii) $\int_{-\infty}^{\infty} K_n(x)dx \to 1$ *as* $n \to \infty$.

(iii) $\int_{\{x||x|>\delta\}} |K_n(x)| \to 0$ *as* $n \to \infty$ *for all* $\delta > 0$.

Show that $K_n * f \to f$ uniformly, where

$$K_n * f(x) = \int_{-\infty}^{\infty} K_n(y)f(x-y)dy.$$

Proof. For all $x \in \mathbb{R}$,

$$\begin{aligned} |K_n * f(x) - f(x)| &\leq \left|K_n * f(x) - \int_{-\infty}^{\infty} K_n(y)f(x)dy\right| + \left|\int_{-\infty}^{\infty} K_n(y)f(x)dy - f(x)\right| \\ &\leq \int_{-\infty}^{\infty} |K_n(y)||f(x-y) - f(x)|dy + \|f\|_\infty \left|\int_{-\infty}^{\infty} K_n(y)dy - 1\right| \\ &\leq \int_{-\infty}^{\infty} |K_n(y)||f(x-y) - f(x)|dy + c\epsilon \\ &= \int_{B(0,\delta)} |K_n(y)||f(x-y) - f(x)|dy + \int_{B(0,\delta)^c} |K_n(y)||f(x-y) - f(x)|dy + c\epsilon \end{aligned}$$

For $\int_{B(0,\delta)} |K_n(y)||f(x-y) - f(x)|dy$, by uniform continuity we ahve

$$\int_{B(0,\delta)} |K_n(y)||f(x-y) - f(x)|dy \leq \epsilon \int_{B(0,\delta)} |K_n(y)|dy \leq \epsilon\|K_n\|_1 < \epsilon M$$

For $\int_{B(0,\delta)^c} |K_n(y)||f(x-y) - f(x)|dy$, by the third assumption we have

$$\int_{B(0,\delta)^c} |K_n(y)||f(x-y) - f(x)|dy \leq 2\|f\|_\infty \int_{B(0,\delta)^c} |K_n(y)|dy \leq 2C\epsilon$$

Let $\epsilon \to 0$, so we've got it. □

Problem 8. *(a) Construct a Lebesgue measurable subset A of $\mathbb{R}$ so that for all reals $a < b$, $0 < m(A \cap [a,b]) < b - a$ where m is Lebesgue measure on $\mathbb{R}$.*

Proof. Enumerate all rational intervals $I_1, I_2, \ldots$. For each I_n, construct a fat Cantor set $N_n \subseteq I_n$ with positive measure.

Since N_n is nowhere dense, there exists some interval $\widetilde{I_n} \subseteq I_n$ and $\widetilde{I_n} \cap N_n = \emptyset$.

Construct another fat Cantor set $M_n \subseteq \widetilde{I_n}$ and define $A := \bigcup M_n$.

Now, for all $I = [a,b]$ there exists some n such that $N_n \subseteq I_n \subseteq I$ with $N_n \cap A = \emptyset$ (can be done by induction). We see $m(A \cap I) \geq m(M_n) > 0$ and

$$m(A \cap I) < m(I \backslash N_n) = m(I) - m(N_n) < m(I) = b - a.$$

□

(b) Suppose $A \subseteq \mathbb{R}$ is a Lebesgue measurable set and assume that

$$m(A \cap (a,b)) \leq \frac{b-a}{2}$$

for any $a, b \in \mathbb{R}$, $a < b$. Prove that $\mu(A) = 0$.

Proof. Consider an open set $U \supseteq A$ with $m(U \backslash A) < \epsilon$. Then $U = \sqcup_{i=1}^{\infty}(a_i, b_i)$ and measurable. So

$$m(U) = m(A \cap U) + m(U \cap A^c) < m(A \cap U) + \epsilon$$

Since

$$m(A \cap U) = m\left(A \cap (\sqcup_{i=1}^{\infty}(a_i,b_i))\right) = \sum_{i=1}^{\infty} m(A \cap (a_i,b_i)) \leq \sum_{i=1}^{\infty} \frac{b_i - a_i}{2} = \frac{1}{2}m(U)$$

then

$$m(U) < \frac{1}{2}m(U) + \epsilon \quad \Rightarrow \quad m(U) < 2\epsilon \quad \Rightarrow \quad m(A) \leq m(U) \to 0$$

□

Problem 9. *Prove or disprove that the unit ball of $L^7(0,1)$ is norm closed in $L^1(0,1)$.*

Proof. Let

$$B := \left\{ f \mid \int_0^1 |f|^7 dx \leq 1 \right\}.$$

Let $\{f_n\} \subseteq B$ such that $f_n \to f$ in L^1. We want to show $f \in B \Leftrightarrow \int_0^1 |f|^7 dx \leq 1$.

Since $f_n \to f$ in L^1, then $f_n \to f$ in measure. Thus, there exists a subsequence f_{n_k} such that $f_{n_k} \to f$ a.e.

Therefore, $|f_{n_k}|^7 \to |f|^7$ a.e.. By Fatou's Lemma,

$$\int_0^1 |f|^7 dx \leq \liminf_k \int_0^1 |f_{n_k}|^7 dx \leq 1.$$

□

Problem 10. *Let C be the Banach space of convergent sequences of real numbers under the supremum norm. Compute the extreme points of the closed unit ball, B, of C and determine whether B is the closed convex hull of its extreme points.*

Proof. If $|x(m)| < 1$ for some m then there exists $\delta > 0$ such that $|x(m) - \delta| \leq 1$, $|x(m) + \delta| \leq 1$.

Define $y_1, y_2 \in B$ such that

$$y_1(n) = x(n) \text{ for } n \neq m \text{ and } y_1(m) = x(m) + \delta$$
$$y_2(n) = x(n) \text{ for } n \neq m \text{ and } y_2(m) = x(m) - \delta$$

Then $y_1 \neq y_2$ and $x = \frac{1}{2}(y_1 + y_2)$ so x is not an extreme point.

If $|x(n)| = 1$ for all n, if $x = \lambda y_1 + (1-\lambda)y_2$ for $y_1 \neq y_2 \in B$ then since $|y_i(n)| \leq n$,

$$|x(n)| = 1 = |\lambda y_1(n) + (1-\lambda)y_2(n)| \leq \lambda |y_1(n)| + (1-\lambda)|y_2(n)| \leq 1$$

Equality holds only when $y_1(n) = y_2(n) = \pm 1$. So $y_1 = y_2$ so x is indeed an extreme point. Also, x needs to be convergent so

$$\text{Ext}(B) = \{x \mid |x(n)| = 1 \ \exists N \text{ s.t. } x(n) = 1 \text{ or } -1 \text{ for all } n > N\}.$$

We note that $B(C)$ is a closed subset of $B(\ell^\infty)$, which is weak*-compact since $(\ell^1)^* = \ell^\infty$. Clearly $B(C)$ is convex, so Krein-Milman implies that $B(C)$ is the weak*-closure of the convex hull of its extreme points. Now any element of a sequence in the weak*-closure is going to be the pointwise limit of a convex combination of ones and negative ones (using the $\delta_n \in \ell^1(\mathbb{N})$ functions for $n \in \mathbb{N}$), so the sequence will be in the norm closure of $B(\ell^\infty)$; it remains to show that such a sequence will be in $B(C)$.

Let $b_n := \sum_{i=1}^n \lambda_i a_n^i$ be the pointwise limit of sequences $(a_n^i) \in \text{Ext}(C)$ where $0 \leq \lambda_i \leq 1$ and $\sum_i \lambda_i = 1$. Fix $\varepsilon > 0$ and pick a finite subset $S \subset \mathbb{N}$ such that $\sum\{\lambda_i : i \in S\} > 1 - \varepsilon$. There exists an $N > 0$ such that $m, n > N \Rightarrow |a_m^i - a_n^i| < \varepsilon$ for $i \in S$ (this would in fact mean $a_m^i = a_n^i$ by characterization of $\text{Ext}(B(C))$ for small enough ε). So

$$|b_m - b_n| \leq \sum_{i \in S} \lambda_i |a_m^i - a_n^i| + \sum_{i \notin S} \lambda_i 2 \leq 3\varepsilon.$$

Therefore (b_n) is Cauchy and hence converges, so (b_n) is in the norm closure of $B(C)$. Hence the weak*-closure and norm closures of $\text{Ext}(B(C))$ coincide. □

Problem 11. *Show that every convex continuous function defined on the convex unit ball of a reflexive Banach space achieves a minimum. (A convex function on a convex subset A of a normed space is a real valued function, f, on A s.t. for every $x, y \in A$ and every $0 < \lambda < 1$ we have $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$.)*

Proof. By Alaoglu, the closed unit ball of a reflexive Banach space is weak-compact (in X)= weak*-compact (in X^{**}).

We take Mazur's theorem without proof: for any convex set A, A is closed iff A is weakly closed. (This is a direct consequence of Hahn-Banach Separation Theorem, which is not in Folland but is an extremely useful theorem. It is worth looking up.)

Let $\alpha = \inf\{f(x) : x \in \overline{B(1,0)}\}$. We look at the following cases:

(1) $\alpha = -\infty$. Let $B_n := f^{-1}((-\infty, -n])$. Then each B_n is nonempty, convex, closed (hence weakly closed) subset of $\overline{B(1,0)}$. These are nested sets with the finite intersection property by assumption, so there exists an $x \in \bigcap_{n=1}^{\infty} B_n$ and $f(x) < -n$ for all n, contradiction.

(2) $\alpha \in \mathbb{R}$. Let $B_n := f^{-1}([\alpha, \alpha + \frac{1}{n}])$. These are nonempty, convex, closed (hence weakly closed) subsets of $\overline{B(1,0)}$. Hence these are nested sets with the finite intersection property, so there exists an $x \in \bigcap_{n=1}^{\infty} B_n$ and $f(x) = \alpha$. □

16 August 2016

Problem 1. *Let $\mathcal{A}$ be the set of all real valued functions on $[0,1]$ for which $f(0) = 0$ and $|f(t) - f(s)|^{1/2} \leq t - s$ for all $0 \leq s < t \leq 1$*

(a) Prove that $\mathcal{A}$ is a compact subset of $C[0,1]$.

Proof. It should be clear to the reader that this question requires Arezela-Ascoli Theorem. To see $\mathcal{A}$ is equicontinuous, fix $x \in [0,1]$ and $\epsilon > 0$. Then for $y \in B(\sqrt{\epsilon}, x)$,

$$|f(x) - f(y)| \leq |x - y|^2 < \epsilon$$

For pointwise bounded, for $x \in [0,1]$ then $|f(x)|^{1/2} = |f(x) - f(0)|^{1/2} \leq x$ implies $|f(x)| \leq x^2$.

To see $\mathcal{A}$ is closed, take a sequence $\{f_n\} \subseteq \mathcal{A}$ such that $f_n \to f$ (i.e. for all open U containing f, there exists N such that for all $n \geq N$, $f_n \in U$), then

$$|f(t) - f(s)| \leq |f(t) - f_n(t)| + |f_n(t) - f_n(s)| + |f_n(s) - f(s)| < 2\epsilon + |t - s|^2$$

This holds for all $\epsilon > 0$ so $|f(t) - f(s)| \leq |t - s|^2$.

Clearly, $f(0) = 0$ so $f \in \mathcal{A}$. Thus $\mathcal{A}$ is closed so by Arzela-Ascoli, $\mathcal{A}$ is compact in $C[0,1]$. □

(b) Prove that $\mathcal{A}$ is a compact subset of $L_1[0,1]$

Proof. Consider the map

$$\begin{aligned} \mathrm{id} : C[0,1] &\to L^1[0,1] \\ f &\mapsto f \end{aligned}$$

Since $\| \mathrm{id} \|_1 = \int_0^1 |f| dx \leq \|f\|_\infty$, id is a bounded map.

From (a), $\mathcal{A}$ is compact in $C[0,1]$ so $\mathrm{id}(\mathcal{A}) = \mathcal{A} \subseteq L^1[0,1]$ is also compact.

Remark: $\mathcal{A}$ is also closed in $L^1[0,1]$ since all compact subsets of a metric space is closed. □

Problem 2. *(a) Let $f(x)$ be a real valued function on the real line that is differentiable almost everywhere. Prove that $f'(x)$ is a Lebesgue measurable function.*

Proof. Let

$$f_n 9x) = \frac{f(x+1/n) - f(x)}{1/n}$$

so $f_n \to f'$ almost everywhere. Since f is differentiable almost everywhere, then f is continuous almost everywhere.

Claim: f is Lebesgue measurable

Let $D = \{\text{all discontinuities of } f\}$ so $m(D) = 0$ and D is measurable. Let $E = D^c = \{x \mid f \text{ is continuous at } x\}$ so E is measurable too.

$$f^{-1}((a,\infty)) = f^{-1}((a,\infty) \cap E) \cup f^{-1}((a,\infty) \cap E^c)$$

Since $f|_E$ is continuous, $f^{-1}(a,\infty) \cap E = f|_E^{-1}(a,\infty)$ is open in E. So $f^{-1}(a,\infty) \cap E = U \cap E$ for some open set $U \subseteq \mathbb{R}$. Then $f^{-1}(a,\infty) \cap E$ is measurable.

Now $f^{-1}(a,\infty) \cap E^c \subseteq E^c$, so completeness implies $f^{-1}(a,\infty) \cap E^c$ is measurable. Thus, f is Lebesgue measurable so the claim holds.

So each f_n is measurable, thus $f' = \lim f_n$ almost everywhere is also Lebesgue measurable. □

(b) Prove that if f is a continuous real valued function on the real line, then the set of points at which f is differentiable is measurable.

Proof. Let

$$F(x,h) = \frac{f(x+h) - f(x)}{h}$$

which is continuous on $\mathbb{R} \times (\mathbb{R} \backslash \{0\})$. If x is a differentiable point of f, then for all $\epsilon > 0$, there exists a $\delta > 0$ and some Y such that for all h with $|h| < \delta$, we have $|F(x,h) - Y| < \epsilon$. i.e.

$$D = \{x \mid \text{differentiable point of } f\} = \bigcap_\epsilon \bigcup_\delta \bigcup_Y \bigcap_{|h|<\epsilon} \{x \mid |F(x,h) - Y| < \epsilon\}$$

For fixed ϵ, δ, Y, h then $\{x \mid |F(x,h) - Y| < \epsilon\}$ is open, thus Borel.

By taking only rational ϵ, δ, Y, h we have D Borel measurable. □

Problem 3. *(a) Let f be a real valued function on the unit interval $[0,1]$. Prove that the set of points at which f is discontinuous is a countable union of closed subsets.*

Proof. f is continuous at p if for all n, there exists an open U containing p such that $|f(x) - f(y)| < 1/n$ for all $x, y \in U$. Fix n and let

$$V_n = \bigcup_p \{p \text{ s.t. there exists an appropriate } U\} = \bigcup \{\text{appropriate } U\}$$

Hence, V_n is open. Then

$$\{\text{points where } f \text{ is continuous}\} = \bigcap_n V_n$$

So $\{\text{points where } f \text{ is discontinuous}\} = \bigcup_n V_n^c$ where V_n^c is closed. □

(b) Prove that there does not exist a real valued function on $[0,1]$ that is continuous at all rational points but discontinuous at all irrational points.

Proof. By (a), the irrational poitns would be a countable union of closed subsets. Note that because any open set in $[0,1]$ contains a rational point, then if $\mathbb{Q}^c_{[0,1]} = \bigcup_n F_n$ where F_n is closed and $F_n^\circ = \emptyset$. Then

$$[0,1] = \mathbb{Q}_{[0,1]} \cup \mathbb{Q}^c_{[0,1]} = \left(\bigcup_{q \in \mathbb{Q}} \{q\} \right) \cup \left(\bigcup_n F_n \right)$$

So $[0,1]$ is a countable union of nowhere dense sets. This contradicts Baire-Category Theorem. □

Problem 4. *Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and let (f_n) be a sequence of measurable functions on X that converges pointwise to zero. Prove that (f_n) converges in measure to zero. Show that the converse is false for $[0,1]$ with Lebesgue measure.*

Proof. Fix $\epsilon > 0$. To show $\mu(\{x \mid |f_n(x)| > \epsilon\}) \to 0$, we need $\forall m \, \exists N_m$ such that $\forall n \geq N_m$, $\mu(\{x \mid f_n(x)| > \epsilon\}) < 1/m$.

By Egoroff's Theorem, there exists some $E \subseteq X$ with $\mu(E) < 1/m$ and $f_n \rightrightarrows 0$ uniformly on E^c.

Thus, $\exists N_m$ such that for $n \geq N_m$ $|f_n(x)| < \epsilon$ for all $x \in E^c$ so

$$\mu(\{x \mid |f_n(x)| > \epsilon\}) \leq \mu(E) < \frac{1}{m} \qquad \forall n \geq N_m$$

Thus, $\mu(\{x \mid |f_n(x)| > \epsilon\}) \to 0$.

Counterexample: Let $f_1 = \chi_{[0,1]}, f_2 = \chi_{[0,1/2]}, f_3 = \chi_{[1/2,1]}, \ldots, f_n = \chi_{[j/2^k,(j+1)/2^k]}$ for $n = 2^k + j, 0 \le j < 2^k$.

So f_n does not approach 0 pointwise, but $f_n \to 0$ in L^1, hence in measure. □

Problem 5. *If f is Lebesgue integrable on the real line, prove that $\lim_{h\to 0} \int_{\mathbb{R}} |f(x+h) - f(x)|dx = 0$.*

Proof. Recall: the set $C_c(\mathbb{R})$ of continuous, compactly supported functions is dense in $L^1(\mathbb{R})$.

Fix $\epsilon > 0$ and find $g \in C_c(\mathbb{R})$ with $\|f - g\|_1 < \epsilon$. Since g is continuous, $\lim_n |g(x + 1/n) - g(x)| = 0$ ofr all x.

Since g is compactly supported, then there exists some compact K such that $\text{supp}(g) \subseteq K$.

So there exists a compact K' such that $\text{supp}(g) \cup \text{supp}(g(x + 1/n)) \subseteq K'$ for all n (this follows from $1/n \ge 1$ for all n since we can take $K' = \{k + x \mid k \in K, x \in [0, 1]\}$).

Dini's theorem implies that $|g(x + 1/n) - g(x)| \rightrightarrows 0$ so

$$\int \mathbb{R}|g(x+1/n) - g(x)|dx = \int_{K'} |g(x + 1/n) - g(x)|dx \to 0$$

So then

$$\begin{aligned} &\int_{\mathbb{R}} |f(x + 1/n) - f(x)|dx \\ \le &\int_{\mathbb{R}} |f(x + 1/n) - g(x + 1/n)|dx + \int_{\mathbb{R}} |g(x + 1/n) - g(x)|dx + \int_{\mathbb{R}} |g(x) - f(x)|dx \\ &< 2\epsilon + \int_{\mathbb{R}} |g(x + 1/n) - g(x)|dx \to 2\epsilon \end{aligned}$$

Since it holds for all $\epsilon > 0$ then $\lim_n \int_{\mathbb{R}} |f(x + 1/n) - f(x)|dx = 0$. □

Problem 6. *Prove or disprove that there exists a sequence (P_n) of polynomials such that $(P_n(t))$ converges to one for every $t \in [0, 1]$ but $\int_0^1 P_n(t)dt$ converges to two as $n \to \infty$.*

Proof. Consider

$$f_n(x) = \begin{cases} n^2 x & x \in [0, 1/n] \\ -n^2 x + 2n + 1 & x \in [1/n, 2/n] \\ 1 & x \in [2/n, 1] \end{cases}$$

(that is, f_n linearly connects the points $(0, 1), (1/n, n + 1), (2/n, 1), (1, 1)$.)

So $f_n(x) \to 0$ for all $x \in [0, 1]$ but $\int_0^1 f_n(x)dx = 2$.

Then by Stone-Weierstrass, we can find polynomials P_n such that $\|f_n - P_n\|_\infty \le 2^{-n}$. Then $\forall x$

$$|P_n(x) - 1| \leq |P_n(x) - f_n(x)| + |f_n(x) - 1| \to 0$$

and $\int_0^1 |f_n(x) - P_n(x)|dx \to 0$ so $\int_0^1 P_n(x)dx \to 2$. □

Problem 7. *Let (f_n) be a uniformly bounded sequence of continuous functions on $[0,1]$ that converges pointwise to zero. Prove that 0 is in the norm closure in $C[0,1]$ of the convex hull of (f_n) (the norm is of course the sup norm on $C[0,1]$).*

Proof. By the Geometrical version of the Hahn-Banach,

$$\overline{\text{conv}\{f_n\}}^{\text{weak}} = \overline{\text{conv}\{f_n\}}^{\|\cdot\|}$$

We just need to show that $0 \in \overline{\text{conv}\{f_n\}}^{\text{weak}}$. By Riesz-Representation Theorem, $C[0,1]^* = \mathcal{M}[0,1]$. For all $\mu \in M[0,1]$,

$$\left| \int_{[0,1]} f_n d\mu \right| \leq \int_{[0,1]} |f_n| d|\mu| \to 0$$

by Dominated Convergence Theorem. Thus, $f_n \to 0$ weakly. □

Problem 8. *Assume that X is a reflexive Banach space and ϕ is a continuous linear functional on X. Prove that ϕ achieves its norm; that is, prove that there is a norm one vector x in X such that $\phi(x) = \|x\|$. Show by example that there is a continuous linear functional on the Banach space ℓ_1 that does not achieve its norm.*

Proof. Recall: X reflexive $\Rightarrow \overline{B_X}$ is weak-compact $\Rightarrow \overline{B_X}$ is weak-sequentially compact.

There exists a sequence $\{x_n\} \subseteq \overline{B_X}$ such that $\phi(x_n) \nearrow \|\phi\|$.

Choose a weakly-convergent subsequence $\{x_{n_k}\}$ that converges to $x \in \overline{B_X}$. Then for all $\varphi \in X^*$, $\varphi(x_{n_k}) \to \varphi(x)$.

In particular,

$$\|\phi\| = \lim_n \phi(x_n) = \lim_k \phi(x_{n_k}) = \phi(x).$$

Alternative Proof. For all $\phi \in X^*$, by Hahn-Banach Separation Theorem, there exists some $x^{**} \in X^{**}$ such that $\|x^{**}\|_{X^{**}} = 1$ and $x^{**}(\phi) = \|\phi\|_{X^*}$.

Since X is reflexive, $\exists x \in X$ such that $\hat{x} = x^{**}$ so

$$\|\phi\|_{X^*} = x^{**}(\phi) = \hat{x}(\phi) = \phi(x).$$

Counterexample: Choose $y = (1 - 1/n)_n \in \ell^\infty$. Then $\forall x \in \ell_1$,

$$y(x) = \left|\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)x(n)\right| \leq \sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)|x(n)| < \sum_{n=1}^{\infty}|x(n)| = \|x\|_1 = 1 = \|y\|_{\infty}.$$

□

Problem 9. *Suppose that X is a non separable Banach space. Prove that there is an uncountable subset A of the unit ball of X such that for all $x \neq u$ in X, $\|x-y\| > 0.9$.*

Proof. By transfinite induction, construct $(x_\alpha)\alpha < \omega_1 \subseteq \overline{B_X}$ where ω_1 is the uncountable ordinal.

Given $\alpha < \omega_1$, let $U_\alpha := \overline{\text{span}\{x_\beta \mid \beta < \alpha\}}$ which is separable.

Since X is not separable, $U_\alpha \subsetneq X$.

By Riesz-Lemma, there exists $\|x_\alpha\| = 1$ such that $d(x_\alpha, U_\alpha) \geq 1 - \epsilon$ (put $\epsilon > 0.1$).

So (x_α) satisfies $\|x_\alpha - x_\beta\| \geq 0.9$ and is uncountable.

Alternative Proof if it were not restricted to B_X. Fix $r > 0$. Zornicate over all subsets $A \subseteq X$ such that $\forall x \neq y$, $\|x-y\| > r$.

Find a maximal subset $A_r \subseteq X$ as above. If A_r is uncountable, by scaling of r, we're done.

Suppose not, so each A_r is countable. Enumerate as $\{x_n^r\}_n$. By maximality, for all $x \in X$, $\forall \epsilon > 0$ if $r > 1/\epsilon$ then there exists $n \in \mathbb{N}$ such that $\|x - x_n^m\| < r < \epsilon$ (i.e. $d(x, A_r) < r, \forall x \in X$).

Let $A = \bigcup_{q \in \mathbb{Q}} A_q$ so A is a countable dense subset of X. Contradiction!

Therefore, there exists $q \in \mathbb{Q}$ such that A_q is uncountable. Consider $A' = \{\frac{0.9}{q}x \mid x \in A_q\}$ so for all $x', y' \in A$,

$$\|x'-y'\| = \left\|\frac{0.9}{q}x - \frac{0.9}{q}y\right\| = \frac{0.9}{q}\|x-y\| > 0.9$$

Thus, there eixsts an uncountable $A \subseteq X$ such that for all $x, y \in A$, $\|x-y\| > 0.9$. □

Problem 10. *If A is a Borel subset of the line, then $E = \{(x,y) \mid x-y \in A\}$ is a Borel subset of the plane. If the Lebesgue measure of A is 0, then the Lebesgue measure of E is 0.*

Proof. Define $f(x,y) = x-y : \mathbb{R}^2 \to \mathbb{R}$. This is continuous. Let

$$\mathcal{A} := \{S \subseteq \mathbb{R} \mid f^{-1}(S) \text{ is a Borel set of } \mathbb{R}^2\}$$

Then $\mathcal{A}$ is a σ-algebra (easy to check). If S is open, then $f^{-1}(S)$ is open in $\mathbb{R}^2$, thus Borel. So $\{\text{open sets}\} \subseteq \mathcal{A}$ and so the Borel algebra is a subset of $\mathcal{A}$. In particular, $A \in \mathcal{A}$.

Let $E = f^{-1}(A)$ which is a Borel set of $\mathbb{R}^2$. If $m(A) = 0$, let

$$E^y = \{x \in \mathbb{R} \mid (x,y) \in E\} = y + A$$

This is a null set since $m(y+A) = m(A) = 0$. Thus, $(m\times m)(E) = \int m(E^y)dm(y) = 0$. □

17 January 2016

Problem 1. *Let E be a measurable subset of $[0,1]$. Suppose there exists $\alpha \in (0,1)$ such that*

$$m(E\cap J) \geq \alpha \cdot m(J)$$

for all subintervals J of $[0,1]$. Prove that $m(E) = 1$.

Proof. It's easy to see that $m(E) \leq 1$.

For any open $U \subseteq [0,1]$, write $U = \sqcup_{i=1}^{\infty} I_i$ where each I_i is an open interval. Then

$$m(E\cap U) = \sum_{i=1}^{\infty} m(E\cap I_i) \geq \sum_{i=1}^{\infty} \alpha m(I_i) = \alpha m(U).$$

Assume $m(E) < 1$, so $m(E^c) := a > 0$. We may find some open $U \supseteq E^c$ such that $m(U\cap E) = m(U\backslash E^c) < \epsilon$. So

$$\epsilon > m(U\cap E) \geq \alpha m(U) \geq \alpha m(E^c) = \alpha a > 0.$$

Letting $\epsilon \to 0$, this leads to a contradiction. □

Problem 2. *Let $f, g \in L^1([0,1])$. Suppose*

$$\int_0^1 x^n f(x)dx = \int_0^1 x^n g(x)dx$$

for all integers $n \geq 0$. Prove that $f(x) = g(x)$ a.e.

Proof. See # 2 from January 2017. □

Problem 3. *Let $f, g \in L^1([0,1])$. Assume for all functions $\varphi \in C^\infty[0,1]$ with $\varphi(0) = \varphi(1)$ we have*

$$\int_0^1 f(t)\varphi'(t)dt = -\int_0^1 g(t)\varphi(t)dt.$$

Show that f is absolutely continuous and $f' = g$ a.e.

Proof. Fix $x \in [0,1]$ and construct h_n via

$$h_n(t) = \begin{cases} nt & t \in [0, 1/n] \\ 1 & t \in [1/n, x] \\ 1 - n(t-x) & t \in [x, x+1/n] \\ 0 & t \in [x+1/n, 1] \end{cases}$$

(i.e. $h_n(t)$ linearly connects the points $(0,0), (1/n, 1), (x, 1), (x+1/n, 0)$, and $(1,0)$.

Since $C^\infty[0,1]$ is dense in $\|\cdot\|_\infty$, we may use this example rather than some $\varphi \in C^\infty[0,1]$ (i.e. pass to the continuous case). Then

$$\int_0^1 f(t)h_n'(t)dt = \int_0^{1/n} f(t)ndt + 0 + \int_x^{x+1/n} f(t)(-n)dt + 0 \to f(0) - f(x)$$

where the limit follows from Lebesgue Differentiation Theorem. Also,

$$\begin{aligned}\int_0^1 g(t)h_n(t)dt &= \int_0^{1/n} nt \underbrace{g(t)}_{\to 0} dt + \int_{1/n}^x g(t)dt + \int_x^{x+1/n} g(t)dt - \int_x^{x+1/n} n \underbrace{(t-x)g(t)}_{\to 0 \text{ as } t \to x} dt + 0 \\ &\to 0 + \int_{1/n}^{x+1/n} g(t)dt - 0\end{aligned}$$

where the limit again follows from Lebesgue Differentiation Theorem. Taking the limit as $n \to \infty$ on both sides, we get $\int_0^x g(t)dt = \lim_n \int_{1/n}^{x+1/n} g(t)dt$. So

$$f(0) - f(x) = \lim_n \int_0^1 f(t)h_n'(t) = \lim_n -\int_0^1 g(t)h_n(t)dt = -\int_0^x g(t)dt$$

Implying $f(x) = f(0) + \int_0^x g(t)dt$. Then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f(0) + \int_0^{x+h} g(t)dt - f(0) - \int_0^x g(t)dt}{h} = \lim_{h \to 0} \frac{\int_x^{x+h} g(t)dt}{h} = g(x)$$

and f is absolutely continuous. □

Problem 4. *Let $\{g_n\}$ be a sequence of measureable functions on $[0,1]$ such that*

(i) $|g_n(x)| \le C$, *for a.e.* $x \in [0,1]$

(ii) and $\lim_{n\to\infty} \int_0^a g_n(x)dx = 0$ *for every* $a \in [0,1]$.

Prove that for each $f \in L^1([0,1])$, we have

$$\lim_{n\to\infty}\int_0^1 f(x)g_n(x)dx = 0.$$

Proof. Let $S = \text{span}\{\chi_{[0,a]} \mid a \in [0,1]\}$. Then S is dense in the space of step functions in L^1. Step function space is dense in L^1 so S is dense in L^1. Then for every $f \in L^1[0,1]$ there exists a sequence $h_m = \sum_{i=1}^{K_m} K_i^{(m)}\chi_{[0,a_i]} \to f$ in L^1.

For a fixed m,

$$\lim_n \int_0^1 h_m g_n dx = \sum_{i=1}^{K_m} K_i^{(m)} \lim_n \int_0^{a_i} g_n(x)dx = 0$$

where the second equality follows from (ii). For every $\epsilon > 0$, we can choose some m such that $\|h_m - f\|_1 < \epsilon$.

For that m, choose some N such that $\left|\int_0^1 h_m g_n\, dx\right| < \epsilon$ for all $n > N$. Then

$$\begin{aligned}\left|\int_0^1 f(x)g_n(x)dx\right| &\le \left|\int_0^1 \big(f(x)-h_m(x)\big)g_n(x)dx\right| + \left|\int_0^1 h_m(x)g_n(x)dx\right|\\ &\le c\|f-h_m\|_1 + \epsilon\\ &< (c+1)\epsilon\end{aligned}$$

Thus, $\int_0^1 f(x)g_n(x)dx = 0.$ □

Problem 5. *(a) Let X be a normed vector space and Y be a closed linear subspace of X. Assume Y is a proper subspace, that is, $Y \neq X$. Show that, for all $0 < \epsilon < 1$, there is an element $x \in X$ such that $\|x\| = 1$ and*

$$\inf_{y\in Y}\|x-y\| > 1-\epsilon$$

Proof. Fix some $x_0 \in X\backslash Y$, denote $\inf_{y\in Y}\|x_0 - y\| = d > 0$. Now for every $\epsilon > 0$ choose some $\delta > 0$ such that $\frac{d}{d+\delta} > 1-\epsilon$.

Choose $y_0 \in Y$ such that $\|x_0 - y_0\| < d+\delta$. Let $x = \frac{x_0-y_0}{\|x_0-y_0\|}$ so $\|x\| = 1$ and

$$\inf_{y\in Y}\|x-y\| = \inf_{y\in Y}\left\|\frac{x_0-y_0}{\|x_0-y_0\|} - y\right\| = \frac{1}{\|x_0-y_0\|}\inf_{y'\in Y}\|x_0-y'\| > \frac{d}{d+\delta} > 1-\epsilon.$$

□

(b) Use part (a) to prove that, if X is an infinite dimensional normed vector space, then the unit ball of X is not compact.

Proof. If we construct a sequence $\{x_n\}$ such that there are no convergent subsequences, we are done.

Assume we have chosen $\{x_1, x_2, \dots, x_{n-1}\} \subseteq \overline{B_X}$. Let $Y = \operatorname{span}\{x_1, x_2, \dots, x_{n-1}\}$. By part (a), there exists some $x_n \in B$ such that $\|x_n\| = 1$ and $\inf_{y \in Y} \|x_n - y\| > 1/2$.

Then we have a sequence $\{x_n\} \subseteq \overline{B_X}$ such that $\|x_n - x_m\| > 1/2$ for all $n \neq m$ so no convergent subsequence may exist. □

Problem 6. *Let $\{f_k\}$ be a sequence of increasing functions on $[0,1]$. Suppose*

$$\sum_{k=1}^{\infty} f_k(x)$$

converges for all $x \in [0,1]$. Denote the limit function by f, that is,

$$f(x) = \sum_{k=1}^{\infty} f_k(x).$$

Prove that

$$f'(x) = \sum_{k=1}^{\infty} f_k'(x), \quad a.e. \ x \in [0,1].$$

Proof. It's easy to see f is increasing, so it's differentiable almost everywhere. Let $F_N = \sum_{n=1}^{N} f_n$ so $F_N \to f$ for all $x \in [0,1]$. Choose an increasing sequence N_k such that $0 \leq f(1) - F_{N_k}(1) \leq 2^{-k}$. Then

$$\sum_{k=1}^{\infty} \big(f(1) - F_{N_k}(1)\big) \leq \sum_{k=1}^{\infty} 2^{-k} = 1.$$

Now, let $g(x) := \sum_{k=1}^{\infty} \big(f(x) - F_{N-k}(x)\big) = \sum_{k=1}^{\infty} \sum_{n=N_k+1}^{\infty} f_n(x)$.

Since $\sum_{n=N_k+1}^{\infty} f_n(x)$ is increasing as x increases, then g is increasing.

So $0 \leq g(x) \leq g(1) \leq 1$ and g is differentiable almost everywhere. Now,

$$\frac{1}{h}\big(g(x+h) - g(x)\big) = \frac{1}{h} \sum_{k=1}^{\infty} \big(f(x+h) - F_{N_k}(x+h)\big) - \big(f(x) - F_{N_k}(x)\big).$$

So since $f'(x) - F_{N_k}(x) = \sum_{n=N_k+1}^{\infty} f_n(x)$ is increasing, $g'(x) \geq \sum_{k=1}^{\infty} f'(x) - F_{N_k}'(x) \geq 0$. Therefore, $\sum_{k=1}^{\infty} f'(x) - F_{N_k}(x)$ converges. So $\lim_k F_{N_k}'(x) = f'(x)$, implying $f'(x) = \sum_{k=1}^{\infty} f_k'(x)$ almost everywhere. □

Problem 7. *Suppose $f, g : [a,b] \to \mathbb{R}$ are both continuous and of bounded variation. Show that the set*

$$\{(f(t), g(t)) \in \mathbb{R}^2 \mid t \in [a,b]\}$$

cannot cover the entire unit square $[0,1]\times[0,1]$.

Proof. Define $r(t)=(f(t),g(t))$. Since $\mathbb{R}^2$ is finite dimensional, $\ell^1\sim\ell^2$. Since f and g have bounded variation, so does r. Thus, we know that whenever $a=x_0<x_1<x_2<\ldots<x_n=b$ we have $\sum_{i=1}^n \|r(x_i)-r(x_{i-1})\|_2<M$.

Now suppose $[0,1]\times[0,1]$ can be covered. Divide $[0,1]\times[0,1]$ into n^2 small squares with center z_j and the length of each edge is $1/n$. Then choose t_j such that $r(t_j)=z_j$.

Now relabel/reorder the t_j in increasing order so that $s_1<s_2<\ldots<s_{n^2}$. Then since the distance between two centers is at least $1/n$,

$$\sum_{j=1}^{n^2-1}\|r(s_{j+1})-r(s_j)\|_2\geq\sum_{j=1}^{n^2-1}1/n=\frac{n^2-1}{n}\to\infty.$$

This is a contradiction! □

Problem 8. *Prove the following two statements:*

(a) Suppose f *is a measurable function on* $[0,1]$, *then*

$$\|f\|_{L^\infty}=\lim_{p\to\infty}\|f\|_{L^p}$$

Proof. In $[0,1]$, by Hölder, we know that $\|f\|_p\leq\|f\|_q$ when $p\leq q$. Also, $\|f\|_p\leq\|f\|_\infty$ for all p. Therefore, $\|f\|_p\nearrow\leq\|f\|_\infty$ and so $\lim_p\|f\|_p\leq\|f\|_\infty$.

On the other hand, for every $\epsilon>0$, let $E=\{x\mid|f(x)|>\|f\|_\infty-\epsilon\}$ and $0<\mu(E)\leq1$ since $\|f\|_\infty=\operatorname{esssup}|f(x)|$. Then $\|f\|_p^p\geq\int_E|f|^p>\left(\|f\|_\infty-\epsilon\right)^p\mu(E)$. Take $p\to\infty$ so $\lim_p\|f\|_p\geq\|f\|_\infty-\epsilon$, implying $\lim_p\|f\|_p\geq\|f\|_\infty$. □

(b) If $f_n\geq0$ *and* $f_n\to f$ *in measure, then* $\int f\leq\liminf\int f_n$.

Proof. Choose a subsequence $\{f_{n_k}\}$ such that $\lim_k\int f_{n_k}=\liminf\int f_n$. Since $f_n\to f$ in measure, $f_{n_k}\to f$ in measure, so there exists a further subsequence $\{f_{n_{k_\ell}}\}\to f$ a.e. Then by Fatou's Lemma,

$$\int f=\int\lim_\ell f_{n_{k_\ell}}\leq\liminf_\ell\int f_{n_{k_\ell}}=\liminf_k f_{n_k}=\liminf_n\int f_n.$$

□

Problem 9. *Suppose* $\{f_n\}$ *is a sequence of functions in* $L^2([0,1])$ *such that* $\|f_n\|_{L^2}\leq1$. *If* f *is measurable and* $f_n\to f$ *in measure, then*

(a) $f\in L^2([0,1])$;

Proof. $f_n \to f$ in measure implies $\{f_{n_k}\} \to f$ almost everwhere which implies $|f_{n_k}|^2 \to |f|^2$ almost everywhere. By Fatou's Lemma,

$$\int_0^1 |f|^2 dx \leq \varliminf_n \int_0^1 |f_{n_k}|^2 dx \leq 1.$$

So $f \in L^2$. □

(b) $f_n \to f$ *weakly in* L^2;

Proof. Let $g \in L^2$. We want to show that $f_n g \to fg$ in L^1. Now, $f_n \to f$ in measure, then $f_n g \to fg$ in measure and thus is Cauchy in measure.

Define $A_{m,n} = \{x \in [0,1] \mid |f_n g(x) - f_m g(x)| \geq \epsilon\}$. Then

$$\int_0^1 |f_n g - f_m g| dx = \int_{A_{m,n}} |f_n g(x) - f_m g(x)| dx + \int_{[0,1]\setminus A_{m,n}} |f_n g(x) - f_m g(x)| dx \leq \int_{A_{m,n}} |f_n g| + |f_m g| dx + \epsilon.$$

We know for all $\epsilon > 0$ there exists some $\delta > 0$ such that $\mu(A_{m,n}) < \delta$,

$$\int_{A_{m,n}} |f_n g| \leq \left(\int_{A_{m,n}} |f_n|^2 dx\right)^{1/2} \left(\int_{A_{m,n}} |g|^2 dx\right)^{1/2} \leq \left(\int_{A_{m,n}} |g|^2\right)^{1/2} < \epsilon.$$

since $g \in L^2$. Then since $\{f_n g\}$ is Cauchy in measure, there exists some N such that for all $m, n > N$, $\mu(A_{m,n}) < \delta$. Then $\int_0^1 |f_n g - f_m g| dx < 3\epsilon$ implies $\{f_n g\}$ is Cauchy in L^1.

Therefore, there exists some $h \in L^1$ such that $f_n g \to h$ in L^1.

We know $f_n g \to fg$ in measure, so $f_{n_k} g \to fg$ almost everywhere. Also, $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\int_A |f_{n_k} g| < \epsilon$ for all A such that $\mu(A) < \delta$.

Therefore, $\{f_{n_k}\}$ is uniformly integrable. By Viteli Convergence Theorem, $f_{n_k} g \to fg$ in L^1. Thus, $h = fg$ so $f_n g \to fg$ in L^1. So $f_n \to f$ weakly.

<u>Note:</u> We could also have used the uniqueness of limit in the measure. □

(c) $f_n \to f$ *with respect to norm in* L^p *for* $1 \leq p < 2$.

Proof. Define $E_n = \{x \mid |f_n(x) - f(x)| \geq \epsilon\}$. From problem 8 on this exam, we know $\|f_n\|_p \leq \|f_n\|_2 \leq 1$ and $\|f\|_p \leq \|f\|_2 < \infty$. Then

$$\int |f_n - f|^p = \int_{E_n} |f_n - f|^p + \int_{E_n^c} |f_n - f|^p dx \leq 2^{p-1} \int_{E_n} |f_n|^p + |f|^p + \epsilon$$

where the inequality follows from the fact that $|a - b|^p \leq 2^{p-1}(|a|^p + |b|^p)$.

Since $f_n \to f$ in measure and $m(E_n) \to 0$ as $n \to \infty$, so since $f \in L^p$ then $\int_{E_n} |f|^p dx \to 0$ as $n \to \infty$.

For some $A \subseteq [0,1]$, we have

$$\int_A |f_n|^p = \int_0^1 |f_n|^p \chi_A \leq \||f_n|^p\|_{2/p} \|\chi_A\|_{2/2-p} = \|f_n\|_2^p m(A)^{\frac{2}{2-p}} \leq m(A)^{\frac{2}{2-p}}.$$

So similar to the previous case, we can take $m(E_n)$ small enough such that $\int_{E_n} |f_n|^p dx < \epsilon$ for any fixed $1 \leq p < 2$. □

There are a few hints in the qual

Problem 10. *Suppose E is a measurable subset of $[0,1]$ with Lebesgue measure $m(E) = \frac{99}{100}$.*

Show that there exists a number $x \in [0,1]$ such that for all $r \in (0,1)$,

$$m(E \cap (x-r, x+r)) \geq \frac{r}{4}.$$

Hint: Use the Hardy-Littlewood maximal inequality

$$m(\{x \in \mathbb{R} \mid Mf(x) \geq \alpha\}) \leq \frac{3}{\alpha}\|f\|_1$$

for all $f \in L^1(\mathbb{R})$. Here Mf denotes the Hardy-Littlewood Maximal function of f.

Proof. The Hardy-Littlewood Maximal function of χ_A is

$$M\chi_A = \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} \chi_A(x)dx = \sup_{r>0} \frac{1}{2r} m(A \cap (x-r, x+r)).$$

Assume the result is not true. Then $\forall x \in [0,1]$, $\exists r_x \in (0,1)$ such that $m(E \cap (x - x_r, x + x_r)) < \frac{r_x}{4}$. This happens if and only if $\frac{1}{2r_x} m(E \cap (x - r_x, x + r_x)) < 1/8$ which is equivalent to $\frac{1}{2r_x} m(E^c \cap (x - r_x, x + r_x)) \geq \frac{7}{8}$.

Now set $A = E^c$ so $M\chi_A(x) \geq \frac{7}{8}$. However,

$$m\big(\{x \in [0,1] \mid M\chi_A(x) \geq \frac{7}{8}\}\big) \leq 3\frac{8}{7}\|\chi_A\| = \frac{24}{7}\frac{1}{100} = \frac{24}{100}.$$

But we need it to be equal to 1. Contradiction! □

18 August 2015

Problem 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. For each $t \in \mathbb{R}$ define*

$$f_t(x) = f(t + x), \quad x \in \mathbb{R}.$$

Prove that $f_t(x)$ is a Borel measurable function (in x) for each fixed $t \in \mathbb{R}$.

Proof. We see that

$$f_t^{-1}(-\infty, a) = \{x \mid f(x+t) \in (-\infty, a)\} = \{x \mid x + t \in f^{-1}(-\infty, a)\} = f^{-1}((-\infty, a)) - t = B - t.$$

Since $T_t(x) = x + t$ is continuous, then $T_t^{-1}(B) = B - t$ is Borel. □

Problem 2. *Justify the statement that*

$$\int_0^1 \int_0^1 \frac{(x-y)\sin(xy)}{x^2+y^2} dx\ dy = \int_0^1 \int_0^1 \frac{(x-y)\sin(xy)}{x^2+y^2} dy\ dx.$$

Proof. We just need to show that $\int_0^1 \int_0^1 \left| \frac{(x-y)\sin(xy)}{x^2+y^2} \right| dxdy < \infty$. But

$$\int_0^1 \int_0^1 \left| \frac{(x-y)\sin(xy)}{x^2+y^2} \right| dxdy = \int_0^{\pi/2} \int_0^{\sqrt{2}} \left| \frac{r\cos\theta - r\sin\theta}{r^2} \right| |r| dr d\theta \leq 2 \int_0^{\pi/2} \int_0^{\sqrt{2}} dr d\theta = \sqrt{2}\pi < \infty.$$

So the function is in L^1 and Fubini gives us the desired result. □

Problem 3. *Assume that (f_n) is a sequence in $C[0,1]$.*

(a) *Show that (f_n) converges weakly to 0 if and only if (f_n) is bounded in $C[0,1]$ and $\lim_{n\to\infty} f_n(t) = 0$ for all $t \in [0,1]$.*

Proof. $\Rightarrow$) We know $C[0,1]^* = \mathcal{M}[0,1]$. Then $f_n \to 0$ weakly implies $\int f_n d\mu \to 0$ for all $\mu \in \mathcal{M}[0,1]$. Choose $\mu = \delta_t$ so

$$\int f_n d\delta_t = f_n(t) \to 0 \quad \forall t \in [0,1]$$

(this follows from the fact that weak convergence implies uniformly bounded). Consider

$$\begin{aligned} \chi : C[0,1] &\to C[0,1]^{**} = \mathcal{M}[0,1]^* \\ \chi(f_n)(\mu) &= \mu(f_n) \end{aligned}$$

Since $\mu(f_n) \to 0$ then $\chi(f_n)(\mu) \to 0$ for all $\mu \in \mathcal{M}[0,1]$. Since convergent sequences are bounded, then $\sup_n |\chi(f_n)(\mu)| \leq M$.

By the uniform boundedness theorem, $\sup_n \|\chi(f_n)\| < \infty$. By isometry, $\|f_n\| = \|\chi(f_n)\|$ so $\sup_n \|f_n\| < \infty$.

$\Leftarrow$) By Dominated Convergence Theorem, $f_n \to 0$ in $L^1(\mu)$. So therefore, $|\int f_n d\mu| \leq \int |f_n| d|\mu| \to 0$. So $f_n \to 0$ weakly. □

(b) *Show that if (f_n) converges weakly in $C[0,1]$, then it converges in norm in $L_p[0,1]$ for all $1 \leq p < \infty$.*

Proof. WLOG $f_n \to 0$ weakly. By (a) we know $f_n(t) \to 0$ and $\|f_n\|_\infty$ is bounded. Thus, $|f_n(t)|^p \to 0$ pointwise and $\||f_n|\|_\infty$ is bounded.

By the Dominated Convergence Theorem, we have $\|f_n\|_p \to 0$. □

Problem 4. *Let A be a Lebesgue null set in $\mathbb{R}$. Prove that*

$$B := \{e^x \mid x \in A\}$$

is also a null set.

Proof. First, assume $A \subseteq [0,1]$. Then $f(x) = e^x$ is Lipschitz-continuous (i.e. $|f(x)-f(y)| \leq M|x-y|$ for some M). Since $m(A) = 0$, we can find $\bigcup_{k=1}^\infty B_k$ where B_k are open intervals such that $A \subseteq \bigcup_{k=1}^\infty B_k$ and $m\left(\bigcup_{k=1}^\infty B_k\right) < \epsilon$. Then

$$m(f(A)) \leq m\left(f\left(\bigcup_{k=1}^\infty B_k\right)\right) \leq \sum_{k=1}^\infty M m(B_k) < M\epsilon.$$

So $m(f(A)) = 0$. Now we can write $A = \bigcup_{n=-\infty}^\infty A \cap [n, n+1]$ so $m(f(A)) = \sum_{-\infty}^\infty m(f(A \cap [n, n+1])) = 0$. □

Problem 5. *(a) Define absolute continuity of a function $f : \mathbb{R} \to \mathbb{R}$ and of a function $f : [a,b] \to \mathbb{R}$.*

Proof. The function $f : [a,b] \to \mathbb{R}$ is absolutely continuous if $\forall \epsilon > 0$, $\exists \delta > 0$ such that whenever a finite sequence of disjoint subintervals $(x_k, y_k) \subseteq I$ satisfies $\sum_{k=1}^N (y_k - x_k) < \delta$ then $\sum_{k=1}^\infty |f(y_k) - f(x_k)| < \epsilon$. □

(b) *Show that if f and g are absolutely continuous on $[a,b]$, $a, b \in \mathbb{R}$, $a < b$, then $f \cdot g$ is absolutely continuous on $[a,b]$.*

Proof. Since f and g are continuous on $[a,b]$, then they achieve a maximum so we can let $M_f = \sup\{f(x) \mid a \leq x \leq b\} < \infty$, $M_g = \sup\{g(x) \mid a \leq x \leq b\}$.

Fix $\epsilon > 0$. Then there exists some $\delta_f, \delta_g > 0$ such that

$$\sum (y_k - x_k) < \delta_f \quad \Rightarrow \quad \sum |f(y_k) - f(x_k)| < \frac{\epsilon}{2M_g}$$
$$\sum (y_k - x_k) < \delta_g \quad \Rightarrow \quad \sum |f(y_k) - f(x_k)| < \frac{\epsilon}{2M_f}$$

Choose finite and disjoint such that $\sum y_k - x_k < \min(\delta_f, \delta_g)$. Then

$$\begin{aligned}\sum|f(y_k)g(y_k)-f(x_k)g(x_k)| &\le \sum|f(y_k)g(y_k)-f(y_k)g(x_k)|+\sum|f(y_k)g(x_k)-f(x_k)g(x_k)|\\ &\le \sum|f(y_k)||g(y_k)-g(x_k)|+\sum|g(x_k)||f(y_k)-f(x_k)|\\ &\le M_f\sum|g(y_k)-g(x_k)|+M_g\sum|f(y_k)-f(x_k)|\\ &\le M_f\frac{\epsilon}{2M_f}+M_g\frac{\epsilon}{2M_g}\\ &=\epsilon\end{aligned}$$

This is what we wanted. □

(c) *Give an example to show that (b) is false if $[a,b]$ is replaced by $\mathbb{R}$.*

Proof. Take $f(x)=g(x)=x$ so $fg=x^2$. Then

$$|(x+\delta)^2-x^2|=|x^2+2\delta x+\delta^2-x^2|=|2\delta x+\delta^2|\to\infty \quad \text{as } x\to\infty.$$

So there does not exist any δ such that $|fg(y)-fg(x)|<\epsilon$ (even for just one interval!) □

Problem 6. *Let X and Y be Banach spaces and $T: X\to Y$ be a one-to-one, bounded and linear operator for which the range $T(X)$ is closed in Y. Show that for each continuous linear functional ϕ on X there is a continuous linear functional ψ on Y, so that $\phi=\psi\circ T$.*

Proof. Since $T: X\to T(X)$ is bijective, by teh open mapping theorem, T^{-1} is bounded so $\phi\circ T^{-1}\in T(X)^*$.

Then by the Hahn-Banach, there exists some $\psi\in Y^*$ such that $\psi(y)=(\phi\circ T^{-1})(y)$ for all $y\in Y$.

For any $x\in X$, $T(x)=y\in Y$ and we have

$$\phi(x)=\phi\big(T^{-1}(Tx)\big)=\psi(Tx)=(\psi\circ T)(x).$$

Since this is true for all $x\in X$, $\phi=\psi\circ T$. □

Problem 7. *State the Open Mapping Theorema nd the Closed Graph Theorem for Banach spaces. Derive the Open Mapping Theorem from the Closed Graph Theorem.*

Proof. Assume $T: X\to Y$ is surjective, linear, and bounded. WLOG we want to show $B(0,\delta)\subseteq T(B(0,1))$ for some $\delta>0$. Define

$$\begin{aligned}G: Y&\to X/\ker(T)\\ y&\mapsto [x]=x+\ker(T) \quad \text{where } y=Tx\end{aligned}$$

Then G is well-defined, because T is surjective.

<u>Claim:</u> G is closed.

Assume $y_n \to y$ in Y and $G(y_n) \to [x]$ in $X/\ker(T)$. WTS $G(y) = [x] \Leftrightarrow Tx = y$.

We have $Tx_n = y_n$ so since $[x_n] \to [x]$ then $\|[x_n] - [x]\| = \inf_{z \in \ker T} \|x_n - x - z\| \to 0$. Then take $(z_n) \subseteq \ker(T)$ such that $\|x_n - x - z_n\| < 1/n$. So $x_n - z_n \to x$. Then

$$\|T(x_n - z_n) - T(x)\| \leq \|T\| \|x_n - x - z_n\| \to 0.$$

Thus, $T(x_n - z_n) = T(x_n) \to T(x)$. And also $T(x_n - z_n) = T(x_n) = y_n \to y$. Together, these imply $T(x) = y$. So G is closed, and the claim holds.

By the closed graph theorem, G is bounded so there exists some $\delta > 0$ such that $G(B(0, \delta)) \subseteq B(0, 1)$ in $X/\ker(T)$. Now, let $y \in B(0, \delta)$ so then $[x] = G(y) \in B(0, 1)$. Thus, if $\inf_{z \in \ker(T)} \|x - z\| < 1$, then there exists some $z_0 \in \ker(T)$ such that $\|x - z_0\| < 1$. This implies $y = Tx = T(x - z_0) \in T(B(0, 1))$ so $B(0, \delta) \subseteq T(B(0, 1))$. □

Problem 8. *Let Y be a closed subspace of a Banach space X, with norm $\|\cdot\|$. Let $\|\cdot\|_1$ be a norm on Y which is equivalent to $\|\cdot\|$, meaning that there is a $C \geq 1$ so that*

$$\frac{1}{C}\|y\|_1 \leq \|y\| \leq C\|y\|_1 \text{ for all } y \in Y.$$

Let S be the set of all linear functions $\phi : X \to \mathbb{R}$, so that

(i) $|\phi(y)| \leq \|y\|_1$ *for all* $y \in Y$, *and*

(ii) $|\phi(x)| \leq C\|x\|$ *for all* $x \in X$.

Prove the following statements

(a) $\|x\|_2 := \sup_{\phi \in S} |\phi(x)|$, $x \in X$, *defines a norm on* X.

Proof. Easy to check. □

(b) $\|y\|_2 = \|y\|_1$ *for* $y \in Y$.

Proof. Since $|\phi(y)| \leq \|y\|_1$ then $\|y\|_2 \leq \|y\|_1$.

On the other hand, from the Hahn-Banach separation theorem, for all $y \neq 0$, there exists some $\phi \in X^*$ such that $\|\phi\| = 1$ and $\phi(y) = \|y\|_1$ so $\|y\|_2 \geq \|y\|_1$.

To check that $\phi \in S$: $|\phi(y)| = \|y\|_1$ and $|\phi(x)| \leq \|\phi\|\|x\| = \|x\|$. □

(c) *The norms* $\|\cdot\|_2$ *and* $\|\cdot\|$ *are equivalent on* X.

Proof. We just need to consider this on $X \backslash Y$. For $x \in X \backslash Y$, we have

$$\|x\|_2 = \sup_{\phi \in S} |\phi(x)| \leq C\|x\|.$$

Again by Hahn-Banach, for $\tilde{x} \neq 0$, there exists some $\phi \in X^*$ such that $\phi(\tilde{x}) = \|\tilde{x}\|$ and $\|\phi\| = 1$. Define $\psi = \frac{1}{C}\phi$ so $\|\psi\| = \frac{1}{C}$.

Then to see $\psi \in S$:

- $|\psi(x)| \leq \frac{1}{C}\|x\| \leq C\|x\|$ for all $x \in X$
- $|\psi(y)| \leq \frac{1}{C}\|y\| \leq \|y\|_1$ for all $y \in Y$

So $\psi \in S$ and $\|\tilde{x}\| \geq |\psi(\tilde{x})| = \frac{1}{C}\|\tilde{x}\|$ so $\frac{1}{C}\|x\| \leq \|x\|_2 \leq C\|x\|$. □

Problem 9. *Let f be increasing on $[0,1]$ and let*

$$g(x) = \limsup_{h\to 0} \frac{f(x+h) - f(x-h)}{2h}, \qquad \text{for } 0 < x < 1.$$

Prove that if $A = \{x \in (0,1) \mid g(x) > 1\}$ then

$$f(1) - f(0) \geq m^*(A).$$

Proof. For $x \in A$,

$$\limsup_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} > 1$$

so for all $\epsilon > 0$, there exists some $h > 0$ such that $2h < \epsilon$ and $\frac{f(x+h)-f(x-h)}{2h} > 1$ if and only if $f(x+h) - f(x-h) > 2h$.

Let $I = \{(x-h, x+h) \mid x \in A, 2h < \epsilon, (x-h, x+h) \subseteq [0,1]\}$. Then I covers A in the sense of Vitali. By Vitali's Lemma, for every $\epsilon > 0$, there exists $I_1, I_2, \ldots, I_n$ disjoint from I such that $m^*\left(A \setminus \bigcup_{i=1}^n I_i\right) < \epsilon$.

Since $m^*(A) = m^*\left(A \setminus \bigcup_{i=1}^n I_i\right) + m^*\left(\bigcup_{i=1}^n I_i\right)$ for all I_i then write $I_i = (x_i - h_i, x_i + h_i)$ and $x_1 - h_1 < x_1 + h_1 < x_2 - h_2 < \ldots < x_n + h_n$. Then

$$m^*(A) < \epsilon + \sum_{i=1}^n 2h_i < \epsilon + \sum_{i=1}^n |f(x_i + h_i) - f(x_i - h_i)|.$$

Since f is increasing, $\sum_{i=1}^n \big(f(x_i+h_i) - f(x_i - h_i)\big) \leq f(1) - f(0)$.

So $m^*(A) < \epsilon + f(1) - f(0)$ so $m^*(A) \leq f(1) - f(0)$. □

Problem 10. *(a) State a version of the Stone-Weierstrass Theorem.*

Proof. See textbook. □

(b) Let A be a uniformly dense subspace of $C[0,1]$ and let

$$B = \left\{ F(x) \mid F(x) = \int_0^x f(t)dt, \quad 0 \le x \le 1, f \in A \right\}.$$

Prove that B is uniformly dense in

$$C_0[0,1] := \{g \in C[0,1] \mid g(0) = 0\}.$$

Proof. Define $B' = \{F(x) \mid F(x) = \int_0^x f(t)dt, 0 \le x \le 1, f \in C[0,1]\}$. First show B is dense in B'.

For every $F \in B'$, $G = B$, $F(x) = \int_0^x f(t)dt$ and $G(x) = \int_0^x g(t)dt$. Then

$$\|F(x) - G(x)\|_\infty \le \int_0^1 |f - g|dt \le \|f - g\|_\infty.$$

Since A is dense in $C[0,1]$, then $\|f - g\|_\infty < \epsilon$ so $\|F - G\|_\infty < \epsilon$. So B is indeed dense in B'.

Then we will show B' is an algebra (in order to use Stone-Weierstrass). Let $F, G \in B'$ so

$$F(x)G(x) = \int_0^x f(t)dt \int_0^x g(s)ds = \int_0^x \int_0^x f(t)g(s)dsdt = \int_0^x F(t)g(t)+G(t)f(t)dt = \int_0^x \int_0^t f(s)g(t)+g(s)f(t)dsdt.$$

Since $F(t)g(t) + G(t)f(t) \in C[0,1]$ then $FG \in B'$.

Also, $x = \int_0^1 1dt \in B'$ so B' separates points.

By Stone-Weierstrass, B' is dense in $C_0[0,1]$ since any function $F \in B'$, $F(0) = 0$. So B is dense in $C_0[0,1]$. □

(c) *Prove that the span of $\{\sin(nx) \mid n \in \mathbb{N}\}$ is dense in $C_0[0,1]$.*

Proof. $\sin(nx) = \int_0^x n\cos(nx)dt$. From part (b), it is sufficient to show

$$A = \text{span}\{n\cos(nx)\} = \text{span}\{\cos(nt)\}$$

is dense in $C[0,1]$. A is an algebra:

- $\cos(nt)\cos(mt) = \frac{1}{2}\big(\cos((m+n)t) + \cos((m-n)t)\big) \in A$
- $\cos(t)$ separates $[0,1]$ (since $1 < \pi/2$) so A is dense in $C[0,1]$.

□

19 January 2015

Problem 1. *Let $f \in L^1(\mathbb{R})$. If*

$$\int_a^b f(x)dx = 0$$

for all rational numbers $a < b$, prove that $f(x) = 0$ for almost all $x \in \mathbb{R}$.

Proof. Let $E^+ := \{x \mid f(x) > 0\}$. Assume $m(E^+) > 0$ (the same argument will show $E^- := \{x \mid f(x) < 0\}$ has measure zero).

There exists some n such that $E^+ \cap [n, n+1]$ has positive measure. Consider F closed in $\mathbb{R}$ and $F \subseteq E^+ \cap [n, n+1]$ with $m(F) > 0$. Then $[n, n+1] \backslash F$ is open in $[n, n+1]$. Thus, $[n, n+1] \backslash F = \bigcup_{n=1}^{\infty} I_n$ for I_n being disjoint open intervals in $[n, n+1]$.

For all $I_n = (a_n, b_n)$, there exists some $(a_{n_i})_i$, $(b_{n_i})_i \subseteq \mathbb{Q}$ such that $a_{n_i} \to a_n$ and $b_{n_i} \to b_n$. Since

$$\int_{a_n}^{b_n} f(x)dx = \int_{\mathbb{R}} f(x)\chi_{[a_n,b_n]}dx \qquad \lim_i f(x)\chi_{[a_{n_i},b_{n_i}]} = f(x)\chi_{[a_n,b_n]}$$

then $|f(x)\chi_{[a_{n_i},b_{n_i}]} \leq |f(x)| \in L^1$ so by Dominated Convergence Theorem, $\int_{a_n}^{b_n} f(x)dx = 0$.

Since $\int_F f(x)dx > 0$, by condition we know $\int_n^{n+1} f(x)dx = 0$ for all n. So then $\int_{[n,n+1]\backslash F} f(x)dx < 0$.

So there exists some $I_m = (a_m, b_m)$ such that $\int_{I_m} f(x)dx < 0$. Contradiction!

Proof #2 as in Problem 3 from August 2014, not restricted to rationals with $f \in L^1$.

For every open U, write $U = \bigcup_{n=1}^{\infty}(a_n, b_n)$ for disjoint open intervals, so

$$\int_U f(x)dx = \sum_{n=1}^{\infty} \int_{a_n}^{b_n} f(x)dx = 0.$$

For every compact $K \subseteq (a, b)$ then $(a, b)\backslash K$ is open in $\mathbb{R}$ and

$$\int_K f(x)dx = \int_a^b f(x)dx - \int_{(a,b)\backslash K} f(x)dx$$

(because each is finite). Suppose $E^+ = \{x \mid f(x) > 0\}$ has positive measure. Since $E^+ = \bigcup_n E_n$ where $E_n = \{x \mid f(x) > 1/n\}$ so there must exist some n such that $m(E_n) > 0$.

By inner regularity, there exists some $K \subseteq E_n$ with $m(K) > 0$. Then

$$0 = \int_K f(x)dx > \int_K \frac{1}{n}dx = \frac{1}{n}m(K) > 0$$

Contradiction! □

Problem 2. *Let $\{g_n\}_{n=1}^{\infty}$ and g be in $L^1(\mathbb{R})$ and satisfy*

$$\lim_{n\to\infty} \|g_n - g\|_1 = 0.$$

Prove that there is a subsequence of $\{g_n\}_{n=1}^{\infty}$ that converges pointwise almost everywhere to g.

Proof. Step 1: Suppose $g_n \to g$ in L^1. Let $E_{n,\epsilon} = \{x \mid |f_n(x) - f(x)| \geq \epsilon\}$. Then

$$\int |f_n - f| \geq \int_{E_{n,\epsilon}} |f_n - f| \geq \epsilon\mu(E_{n,\epsilon})$$

So then $\mu(E_{n,\epsilon}) \leq \frac{1}{\epsilon}\int |f_n - f| \to 0$.

Step 2: We will show that if $g_n \to g$ in measure, then there exists a subsequence that converges to g pointwise almost everywhere.

Suppose for every $\epsilon > 0$, $\mu(\{x \mid |f_n(x) - f(x)| \geq \epsilon\}) \to 0$. Choose a subsequence $\{g_{n_k}\}$ such that if

$$E_j = \{x \mid |g_{n_j}(x) - g_{n_{j+1}}(x)| > 2^{-j}\}$$

satisfies $\mu(E_j) < 2^{-j}$. Let $F_k = \bigcup_{j=k}^{\infty} E_j$ so $\mu(F_k) \leq \sum_{j=k}^{\infty} 2^{-j} \leq 2^{1-k}$. Let $F = \bigcap_k F_k$ so $\mu(F) = 0$. For $x \notin F_k$ and for $i \geq j \geq k$ then

$$|g_{n_i}(x) - g_{n_j}(x)| \leq \sum_{\ell=j}^{i-1} |g_{n_\ell}(x) - g_{n_{\ell+1}}(x)| \leq \sum_{\ell=j}^{i-1} 2^{\ell} \leq 2^{-j} \to 0 \quad \text{as } k \to \infty.$$

So g_{n_k} is pointwise Cauchy on $x \notin F$, so let

$$f(x) = \begin{cases} \lim g_{n_k}(x) & x \notin F \\ 0 & \text{otherwise} \end{cases}$$

So $g_{n_k} \to f$ almost everywhere and $g_n \to f$ in measure since

$$\mu(\{x \mid |g_n(x) - f(x)| \geq \epsilon\}) \leq \underbrace{\mu(\{x \mid |g_n(x) - g_{n_\ell}(x)| \geq \epsilon/2\})}_{\to 0} + \underbrace{\mu(\{x \mid |g_{n_\ell}(x) - f(x)| \geq \epsilon\})}_{\to 0}$$

and

$$\mu(\{x \mid |f(x) - g(x)| \geq \epsilon\}) \leq \underbrace{\mu(\{x \mid |f(x) - g_n(x)| \geq \epsilon/2\}}_{\to 0} + \underbrace{\mu(\{x \mid |g_n(x) - g(x)| \geq \epsilon/2\})}_{\to 0}$$

so $f = g$ almost everywhere. Thus, $\{g_{n_k}\}$ converges to g almost everywhere. □

Problem 3. *Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $\mathcal{N} \subseteq \mathcal{M}$ be a σ-algebra. If $f \geq 0$ is $\mathcal{M}$-measurable and μ-integrable then prove that there exists an $\mathcal{N}$-measurable and μ-integrable function $g \geq 0$ so that*

$$\int_E g d\mu = \int_E f d\mu, \quad E \in \mathcal{N}.$$

Proof. Define $\nu(E) = \int_E f d\mu$ a finite positive measure on $(X, \mathcal{N}, \mu)$. Then since $\mu(E) = 0$, $\nu(E) = 0$ so $\nu \ll \mu$.

Then by Radon-Nikodym Theorem, there exists some $g : X \to [0, \infty)$ and $\mathcal{N}$-measurable and $g \in L^1(\mu)$ such that $\nu(E) = \int_E g d\mu$. Then

$$\nu(E) = \int_E f d\mu = \int_E g d\mu \quad \forall E \in \mathcal{N}.$$

Note: Folland doesn't mention positive but there are other versions that give positive. □

Problem 4. *(a) State the closed graph theorem.*

Proof. See wikipedia. □

(b) If $\mathcal{H}$ is a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ is a linear operator satisfying

$$\langle Tx, y\rangle = \langle x, Ty\rangle, \quad x, y \in \mathcal{H},$$

then prove that T is bounded.

Proof. Let $x_n \to x$ and $Tx_n \to y$. We want to show $Tx = y$.

$$\underbrace{\langle Tx_n, z\rangle}_{\to\langle y,z\rangle} = \langle x_n, Tz\rangle \to \langle x, Tz\rangle = \langle Tx, z\rangle.$$

So $\langle Tx - y, z\rangle = 0$ for all $z \in \mathcal{H}$ so then $Tx - y = 0$ and so $Tx = y$. □

Problem 5. *Let $f, g \in L^1(\mathbb{R})$. Prove that $h \in L^1(\mathbb{R})$, where $h(x)$ is defined by*

$$h(x) = \int_{\mathbb{R}} f(y)g(x - y)dy$$

whenever this integral is finite.

Proof. We want to show that $\int_{\mathbb{R}} |h(x)|dx = \int_{\mathbb{R}} \left|\int_{\mathbb{R}} f(y)g(x - y)dy\right| dx < \infty$. Indeed,

$$\begin{aligned}\int_{\mathbb{R}}\left|\int_{\mathbb{R}} f(y)g(x-y)dy\right| dx &\le \int_{\mathbb{R}}\int_{\mathbb{R}} |f(y)||g(x-y)|dydx\\ &= \int_{\mathbb{R}} |f(y)|\left(\int_{\mathbb{R}} |g(x-y)|dx\right) dy\\ &= \int_{\mathbb{R}} |f(y)|\|g\|_1 dy\\ &= \|g\|_1\int_{\mathbb{R}} |f(y)|dy\\ &= \|g\|_1\|f\|_1 < \infty.\end{aligned}$$

□

Problem 6. *Let $f, g \in C[0,1]$ with $f(x) < g(x)$ for all $x \in [0,1]$.*

(a) Prove that there is a polynomial $p(x)$ so that

$$f(x) < p(x) < g(x), \quad x \in [0,1].$$

Proof. Let $\epsilon = \inf\{g(x) - f(x) \mid x \in [0,1]\}$. Since $h(x) = g(x) - f(x) > 0$ on $[0,1]$ and attains a minimum on the compact set $[0,1]$ then the inf is attained and thus is positive. So $\epsilon > 0$.

By Stone-Weierstrass, polynomials are dense in $C[0,1]$ so there exists a polynomial $p(x)$ such that $\left\|p - \left(\frac{f+g}{2}\right)\right\|_\infty < \epsilon/2$. Then

$$p(x) < \frac{f(x)+g(x)}{2} + \frac{\epsilon}{2} \le \frac{1}{2}\big(f(x)+g(x)+(g(x)-f(x))\big) = \frac{1}{2}(2g(x)) = g(x).$$

$$p(x) > \frac{f(x)+g(x)}{2} - \frac{\epsilon}{2} > \frac{1}{2}\big(f(x)+g(x)-(g(x)-f(x))\big) = \frac{1}{2}(2f(x)) = f(x).$$

So $f(x) < p(x) < g(x)$.

Remark: Let $M = \max\{g(x) - f(x)\}$ then

$$|g(x) - f(x)| \le \left|g(x) - \left(\frac{f+g}{2}\right)(x)\right| + \left|\left(\frac{f+g}{2}\right)(x) - p(x)\right| < \frac{M}{2} + \epsilon.$$

Alternative Proof. Let $M := \min_{x\in[0,1]} g(x) - f(x)$. By Stone-Weierstrass, there exists some $\tilde{p}(x)$ polynomial such that $\|\tilde{p}(x) - g(x)\|_\infty < M/3$. Let $p(x) = \tilde{p}(x) - M/2$. Then

$$g(x) - p(x) = g(x) - \tilde{p}(x) + \frac{M}{2} > \frac{-M}{3} + \frac{M}{2} = \frac{M}{6} > 0.$$

$$p(x)-f(x) = p(x)-g(x)+g(x)-f(x) = \tilde{p}(x)-g(x)-\frac{M}{2}+\big(g(x)-f(x)\big) > \frac{-M}{3}-\frac{M}{2}+M = \frac{M}{6} > 0.$$

So $f(x) < p(x) < g(x)$. □

(b) Prove that there is an increasing sequence of polynomial $\{p_n(x)\}_{n=1}^{\infty}$ *so that*

$$f(x) < p_n(x) < g(x), \quad x \in [0,1],$$

and $p_n \to g$ *uniformly on* $[0,1]$.

Proof. Find p_1 such that $g - \frac{1}{2} < p_1 < g$ with $\left\|p_1 - \left(\frac{g+g-1}{2}\right)\right\| < \frac{1}{4}$. Then $|g(x) - p_1(x)| < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

Recursively find a polynomial p_n such that $p_{n-1} < p_n < g$ with $\left\|p_n - \left(\frac{g+p_{n-1}}{2}\right)\right\| < \frac{1}{2^{n-1}}$, implying

$$|g(x) - p_n(x)| < \frac{M_{n-1}}{2} + \frac{1}{2^{n+1}} = \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} = \frac{1}{2^n}.$$

So $M_n < \frac{1}{2^n}$. Then for every $\epsilon > 0$ choose N such that $\frac{1}{2^N} < \epsilon$, so for all $n > N$

$$|p_n(x) - g(x)| < \frac{1}{2^N} < \epsilon \qquad \forall x \in [0,1].$$

Alternative Proof. From part (a) we can find $f(x) < p_1(x) < g(x)$. Repeating, we can find $p_1(x) < p_2(x) < g(x)$. By requiring ϵ_n instead of M, in $\|\tilde{p}(x) - g(x)\|_\infty < \epsilon$ and letting $\epsilon_n \to 0$, we get

$$\|p_n(x) - g(x)\|_\infty \leq \|p_n(x) - \tilde{p_n}(x)\|_\infty + \|\tilde{p_n}(x) - g\|_\infty < \frac{\epsilon_n}{2} + \frac{\epsilon_n}{3} = \frac{5}{6}\epsilon_n \to 0.$$

□

Problem 7. *If* $f \in L^2(\mathbb{R})$, $g \in L^3(\mathbb{R})$, *and* $h \in L^6(\mathbb{R})$ *then prove that the product* fgh *is in* $L^1(\mathbb{R})$.

Proof. Note: $\left\||f|^k\right\|_p = \left(\int |f|^{kp} dx\right)^{1/p} = \left(\int |f|^{kp} dx\right)^{\frac{1}{kp}p} = \|f\|_{kp}^p$.

Then it follows that

$$\|fgh\|_1 \leq \|f\|_2 \|gh\|_2 \leq \|f\|_2 \||g|^2|h|^2\|_1^{1/2} \leq \|f\|_2 \left(\||g|^2\|_{p=3/2} \||h|^2\|_{q=3}\right)^{1/3} \leq \|f\|_2 \left(\|g\|_3 \|h\|_6\right)^{1/3} < \infty.$$

Where we use $p = 3/2$, $q = 3$ so $\frac{1}{p} + \frac{1}{q} = \frac{2}{3} + \frac{1}{3} = 1$. □

Problem 8. *(a) A point* y *in a metric space* Y *is isolated if the set* $\{y\}$ *is both open and closed in* Y. *Prove that* $y \in Y$ *is isolated if and only if the complement* $\{y\}^C$ *is not dense in* Y.

Proof. $\Rightarrow$) If y is isolated, then $\{y\}$ is open. But $\{y\}^c \cap \{y\} = \emptyset$ so $\{y\}^c$ is not dense.

$\Leftarrow$) Trivially, $\{y\}$ is closed since we're in a metric space. Suppose $\{y\}^c$ is not dense in Y. Then there exists an open $U \neq \emptyset$ such that $U \cap \{y\}^c = \emptyset$ (since A is dense in $Y \Leftrightarrow$ for all open $U \neq \emptyset$, $U \cap A \neq \emptyset$).

But if $U \cap \{y\}^c = \emptyset$ then $U \subseteq \{y\}^{cc} = \{y\}$ so $U = \{y\}$ is open. □

(b) Let X be a countable nonempty complete metric space. Prove that the set of isolated points is dense in X.

Proof. Let $Y \subseteq X$ be the set of isolated points. Let $X \backslash Y = \{z_j\}_{j=1}^{\infty}$ (or $\{z_j\}_{j=1}^{n}$).

Since the singleton $\{z_k\}$ is not an isolated point, by (a) we know $\{z_k\}^c$ is dense in X, so $\overline{\{z_k\}^c} = X$. So each $\{z_k\}^c$ is open and dense in X.

By Baire-Category, $Y = \bigcap_{j=1}^{\infty}\{z_j\}^c$ (or $\bigcap_{j=1}^{n}\{z_j\}^c$) is also dense in Y. □

Problem 9. *Suppose that $f \in L^p(\mathbb{R})$ for all $p \in (1,2)$ and that $\sup_{p\in(1,2)} \|f\|_p < \infty$. Prove that $f \in L^2(\mathbb{R})$ and that*

$$\lim_{p\to 2-} \|f\|_p = \|f\|_2.$$

Proof. Let $A = \{x \mid |f(x)| \geq 1\}$, $B = \{x \mid |f(x)| < 1\}$. Then by Monotone Convergence Theorem, $\int_A |f|^p dx \nearrow \int_A |f|^2 dx$.

Let $p_n \uparrow 2$. WLOG assume $p_1 = 3/2$. We know on B, $|f|^p \leq |f|^{3/2} \in L^1(B)$. By Dominated Convergence Theorem, $\int_B |f|^p dx \to \int_B |f|^2 dx$ which implies $\int_{\mathbb{R}} |f|^p dx \to \int_{\mathbb{R}} |f|^2 dx$.

Therefore, $\|f\|_p^p \to \|f\|_2^2$ so $\|f\|_p^{p/2} \to \|f\|_2$ as $p \to 2$.

Also, since $M = \sup_{p\in(1,2)} \|f\|_p < \infty$, then $\|f\|_p^{p/2-1} \leq M^{p/2-1}$ for all $p \in (1,2)$.

Then $M^{p/2-1} \to 1$ as $p \to 2$ which implies $\|f\|_p^{p/2} - \|f\|_p \to 0$ as $p \to 2$.

So then, $\|f\|_p \to \|f\|_2$ as $p \to 2$ and $\|f\|_2 < \infty$ since $M < \infty$. □

Problem 10. *Let $(X, \|\cdot\|)$ be a normed vector space with a subspace Y and let $\|\cdot\|_1$ be another norm on Y that satisfies*

$$\frac{1}{K}\|y\|_1 \leq \|y\| \leq K\|y\|_1, \quad y \in Y,$$

where $K > 1$ is a fixed constant. Define S to be the set of linear functionals $\phi : X \to \mathbb{R}$ satisfying

(i) $|\phi(y)| \leq \|y\|_1$, $y \in Y$,

(ii) $|\phi(x)| \leq K\|x\|$, $x \in X$.

Prove the following statements:

(a) $\|x\|_2 := \sup\{|\phi(x)| \mid \phi \in S\}$ defines a norm on X.

Proof. See August 2015 □

(b) For $y \in Y$, $\|y\|_1 = \|y\|_2$.

Proof. See August 2015 □

(c) The norms $\|\cdot\|$ and $\|\cdot\|_2$ are equivalent on X.

Proof. See August 2015 □

20 August 2014

Problem 1. *For $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ be continuous, and assume that for every $x \in [0,1]$ the sequence $(f_n(x))$ is decreasing. Suppose that f_n converges pointwise to a continuous function f. Show that this convergence is uniform.*

Proof. WLOG: by replacing f_n by $f_n(x) - f(x)$, these are still continuous and decreasing pointwise. So we want to prove $f_n \rightrightarrows 0$.

This is precisely Dini's Theorem (aka freebie question).

Fix $\epsilon > 0$ and let $U_n = f_n^{-1}((-1,\epsilon)) = \{x \in X \mid g_n(x) < \epsilon\}$ which is open. Then for all x, $f_n(x) \searrow 0$ so there exists N such that for all $n \geq N$, $|f_n(x)| < \epsilon$ which implies $x \in U_n$.

So $[0,1] = \bigcup_n U_n$. By compactness of $[0,1]$, there exists a finite subcover $U_{n_1}, U_{n_2}, \ldots, U_{n_k}$ for $n_1 < n_2 < \ldots < n_k$ but since $U_n \subseteq U_{n+1}$ then $U_{n_1} \subseteq U_{n_2} \subseteq \ldots \subseteq U_{n_k}$.

Therefore, $[0,1] \subseteq U_{n_k} =: U_N$ so for all $x \in [0,1]$, then $x \in f_N^{-1}((-1,\epsilon)) \Leftrightarrow |f_N(x)| < \epsilon$.

Decreasing f_n implies that for all $n \geq N$, $|f_n(x)| < \epsilon$ for all $x \in [0,1]$. □

Problem 2. *Let $f \in L^1(0,\infty)$. For $x > 0$, define*

$$g(x) = \int_0^\infty f(t)e^{-tx}dt.$$

Prove that $g(x)$ is differentiable for $x > 0$ with derivative

$$g'(x) = \int_0^\infty -tf(t)e^{-tx}dt.$$

Proof. Since

$$\int_0^\infty \int_0^x |tf(t)e^{-ty}|dydt = \int_0^\infty t|f(t)|\left(\int_0^x e^{-ty}dy\right)dt = \int_0^\infty \underbrace{-e^{-tx}}_{\leq 1}|f(t)|dt + \int_0^\infty |f(t)|dt \leq 2\int_0^\infty |f(t)|dt < \infty.$$

By Fubini, $h(x) = \int_0^\infty \int_0^x -tf(t)e^{-ty}dtdy = \int_0^\infty f(t)e^{-tx}dt + c$.

So $h(x) = g(x) + c$.

From the definition of h, we know $h'(x) = g'(x)$ and thus $g(x)$ is differentiable. And h is differentiable since it's absolutely continuous. □

Problem 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function such that*

$$\int_a^b f(x)dx = 0 \text{ for every } a < b.$$

Show that $f(x) = 0$ for almost every $x \in \mathbb{R}$.

Proof. See question 1 from January 2015. □

Problem 4. *Let f be Lebesgue measurable on $[0,1]$ with $f(x) > 0$ a.e. Suppose (E_k) is a sequence of measurable sets in $[0,1]$ with the property that $\int_{E_k} f(x)dx \to 0$ as $k \to \infty$.*

Prove that $m(E_k) \to 0$ as $k \to \infty$.

Proof. Let $F_m = \{x \mid f(x) \geq 1/m\}$ so $F_n \subseteq F_{n+1}$.

Since $f(x) > 0$ almost everywhere, then

$$m\left(\bigcup_{n=1}^{\infty} F_n\right) = \lim_n m(F_n) = 1.$$

Fix $\epsilon > 0$, so there exists N such that $m(F_n^c) < \epsilon/2$ for $n \geq N$. Now,

$$\frac{1}{N} m(E_k \cap F_N) \leq \int_{E_k \cap F_N} f(x)dx \leq \int_{E_k} f(x)dx \to 0 \text{ as } k \to \infty.$$

So there exists some K such that $m(E_k \cap F_N) < \epsilon/2$ for all $k \geq K$. Thus,

$$m(E_k) = m(E_K \cap F_N) + m(E_k \cap F_N^c) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad \forall k \geq K.$$

□

Problem 5. *Let (f_n) be a sequence of continuous functions on $[0,1]$ such that for each $x \in [0,1]$ there is an $N = N_x$ so that*

$$f_n(x) \geq 0 \text{ for all } n \geq N_x.$$

Show that there is an open nonempty set $U \subset [0,1]$ and an $N \in \mathbb{N}$, so that $f_n(x) \geq 0$ for all $n \geq N$ and all $x \in U$.

Proof. Let

$$E_n := \{x \mid f_m(x) \geq 0 \ \forall m \geq n\} = \bigcap_{n=m}^{\infty} \{x \mid f_n(x) \geq 0\}$$

so E_n is closed and $E_n \supseteq E_{n+1}$. For all $x \in [0,1]$ there exists $N = N_x$ such that $f_m(x) \geq 0$ for all $m \geq N$. Thus, $x \in E_N$.

Then, $[0,1] = \bigcup_{n=1}^{\infty} E_n$. Since $[0,1]$ is compact, by Baire-Category we know there exists N such that $\overline{E_N}^{\circ} \neq \emptyset$ (i.e. $E_N^{\circ} \neq \emptyset$).

Let $U = E_N^{\circ}$ be open, non-empty so for all $x \in U$, $f_n(x) \geq 0$ for all $n \geq N$. □

Problem 6. *(a) Define the w^*-topology on the dual X^* of a Banach space X.*

Proof. See wikipedia! □

(b) Let X be an infinite dimensional Banach space. What is the w^*-closure of

$$S_{X^*} = \{x^* \in X^* \mid \|x^*\| = 1\}?$$

(as usual, prove your answer.)

Proof. Claim: $\overline{S_{X^*}}^{w^*} = B_{X^*}$.

We know for any $x_1, x_2, \ldots, x_n \in X$, there exists some $x_0^* \neq 0$ such that $x_0^*(x_i) = 0$. Indeed, if this were not true then otherwise, $x_0^*(x_i) \neq 0$ for some i, let $\varphi : X^* \to \mathbb{R}^n$ be $\varphi(x^*) = (x^*(x_1), \ldots, x^*(x_n))$ then φ is injective so $\dim(X^*) \leq \dim(\mathbb{R}^n) = n$. Contradiction, so true.

Now for any $x^* \in B_{X^*}$, consider it's neighborhood (open under the w^*-neighborhood)

$$V = \bigcap_{i=1}^{n} \{y^* \in X^* \mid |\hat{x}_i(x^* - y^*)| = |x^*(x_i) - y^*(x_i)| < \epsilon\}$$

for each $\{x_i\}_{i=1}^n$ choose such an $x_0^* \neq 0$ from the claim.

Consider the line $\{x^* + tx_0^* \mid t \in \mathbb{R}\}$ in X^*.

Since for any $\hat{x}_i$,

$$\hat{x}_i(x^* + tx_0^* - x^*) = t\hat{x}_i(x_0^*) = tx_0^*(x_i) = 0 < \epsilon.$$

Then $\{x^* + tx_0^* \mid t \in \mathbb{R}\} \subseteq V$. Since $\|x^* + tx_0^*\|$ is continuous about t, then we can find t_0 such that $\|x^* + t_0 x_0^*\| = 1 \Rightarrow V \cap S_{X^*} \neq \emptyset$.

Since any neighborhood of x^* contains a neighborhood of the form V as above (i.e. these V's are a neighborhood basis) then $B_{X^*} \subseteq \overline{S_{X^*}}^{w^*}$.

On the other hand, for any $x_0^* \in B_{X^*}$, by Hahn-Banach separation Theorem, we know there exists $x \in X$ and $c \in \mathbb{R}$ such that $x^*(x) < c < x_0^*(x)$ for all $x^* \in B_{X^*}$.

Then for all $\{x_n^*\} \subseteq B_{X^*}$, $x_n^*(x) \leq c < x_0^*(x)$. Therefore, x_0^* isn't an accumulation point of B_{X^*} which implies $\overline{B_{X^*}}^{w^*} = B_{X^*}$. Thus, $\overline{S_{X^*}}^{w^*} \subseteq \overline{B_{X^*}}^{w^*} = B_{X^*}$ so $B_{X^*} = \overline{S_{X^*}}^{w^*}$. □

Problem 7. *(a) State the Riesz Representation Theorem for the dual $L_p^*(\mu)$ of $L_p(\mu)$, $1 \leq p < \infty$.*

Proof. See Wikipedia! □

(b) Let μ be a finite measure on the measurable space (Ω, Σ). Prove the following part of the above theorem:

If $F \in L_p^*(\mu)$, then there exists an $h \in L_1(\mu)$ so that

$$F(\chi_A) = \int_A h d\mu \text{ for all } A \in \Sigma.$$

Proof. Let $\nu(A) = F(\chi_A)$. The goal is to show ν is a σ-finite signed measure.

(a) $\nu(\emptyset) = F(\chi_\emptyset) = F(0) = 0$

(b) Let $\{E_i\}$ be disjoint, let $E = \bigcup_{i=1}^{\infty} E_i$. Then

$$\begin{aligned}
\nu(E) - \sum_{i=1}^{n} \nu(E_i) &= F(\chi_E) - F\left(\sum_{i=1}^{n} \chi_i\right) \\
&\leq \|F\|_{L_p^*} \left\| \chi_E - \sum_{i=1}^{n} \chi_i \right\|_p \\
&\leq \|F\|_{L_p^*} \left\| \sum_{i=n+1}^{\infty} \chi_i \right\|_p \\
&= \|F\|_{L_p^*} \mu\left(\bigcup_{i=n+1}^{\infty} E_i\right)^{1/p} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Therefore, $\nu(E) = \sum_{i=1}^{\infty} \nu(E_i)$.

When $\mu(A) = 0$, then

$$\nu(A) = F(\chi_A) \leq \|F\|_{L_p^*} \|\chi_A\|_p = \|F\|_{L_p^*} \mu(A)^{1/p} = 0.$$

So $\nu \ll \mu$.

Then from the Radon-Nikodyn Theorem, there exists some $h \in L^1(\mu)$ such that $\nu(A) = \int_A h d\mu$. So

$$F(\chi_A) = \nu(A) = \int_A h d\mu.$$

□

Problem 8. *Assume that (x_n) is a weakly converging sequence in a Hilbert space $\mathcal{H}$. Show that there is a subsequence (y_n) of (x_n) so that*

$$\frac{1}{n} \sum_{j=1}^{n} y_j$$

converges in norm.

Proof. WLOG $x_n \to 0$ weakly ($\langle x_n, y\rangle \to \langle 0, y\rangle$ for all $y \in \mathcal{H}$) and we know $\|x_n\|$ is bounded, $\sup_n \|x_n\| \le C$. For $n > m$,

$$\begin{aligned}\left\|\frac{1}{m}\sum_{j=1}^m y_j - \frac{1}{n}\sum_{j=1}^n y_j\right\|^2 &= \left\langle \frac{1}{m}\sum_{j=1}^m y_j - \frac{1}{n}\sum_{j=1}^n y_j, \frac{1}{m}\sum_{j=1}^m y_j - \frac{1}{n}\sum_{j=1}^n y_j\right\rangle \\ &= \left\langle \left(\frac{1}{m}-\frac{1}{n}\right)\sum_{j=1}^m y_j - \frac{1}{n}\sum_{j=m+1}^n y_j, \left(\frac{1}{m}-\frac{1}{n}\right)\sum_{j=1}^m y_j - \frac{1}{n}\sum_{j=m+1}^n y_j\right\rangle \\ &\le \left(\frac{1}{m}-\frac{1}{n}\right)^2\left\|\sum_{j=1}^m y_j\right\|^2 + 2\left|\left\langle \left(\frac{1}{m}-\frac{1}{n}\right)\sum_{j=1}^m y_j, \frac{1}{n}\sum_{j=m+1}^n y_j\right\rangle\right| + \left(\frac{1}{n}\right)^2\left\|\sum_{j=m+1}^n y_j\right\|^2.\end{aligned}$$

Now by induction, we can choose y_j such that $|\langle y_j, \sum_{n=1}^m y_n\rangle| < 2^{-j}$ for all $m \le j-1$. Pick y_1 randomly.

Since $\langle x_n, y\rangle \to \langle 0, y\rangle$ for all $y \in \mathcal{H}$, then we can find y_2 such that $\langle y_2, y_1\rangle < 2^{-2}$.

Similarly, find y_3 such that $\langle y_3, y_1 + y_2\rangle < 2^{-3}$ and $\langle y_3, y_1\rangle < 2^{-3}$, etc. Then

$$\left\|\sum_{j=1}^m y_j\right\|^2 = \left\langle \sum_{j=1}^m y_j, \sum_{j=1}^m y_j\right\rangle = \langle y_m, y_m\rangle + \langle y_m, \sum_{j=1}^{m-1} y_j\rangle + \ldots + \langle y_1, y_1\rangle \le \sum_{j=1}^m \|y_j\|^2 + \sum_{j=1}^m 2^{-j} < \sum_{j=1}^m \|y_j\|^2 + 2.$$

Therefore,

$$\left(\frac{1}{m}-\frac{1}{n}\right)^2\left\|\sum_{j=1}^m y_j\right\|^2 \le \frac{1}{m^2}\left(\sum_{j=1}^m \|y_j\|^2 + 2\right) < \frac{1}{m^2}(mc+2) \to 0 \text{ as } m \to \infty.$$

Similar argument holds for $\left(\frac{1}{n}\right)^2\left\|\sum_{j=m+1}^n y_j\right\|^2$.

Finally,

$$\begin{aligned}\left|\left\langle \left(\frac{1}{m}-\frac{1}{n}\right)\sum_{j=1}^m y_j, \frac{1}{n}\sum_{j=m+1}^n y_j\right\rangle\right| &\le \frac{1}{n}\left(\frac{1}{m}-\frac{1}{n}\right)\sum_{j=m+1}^n \left\langle y_j, \sum_{k=1}^m y_k\right\rangle \\ &\le \frac{1}{n}\left(\frac{1}{m}-\frac{1}{n}\right)\sum_{j=m+1}^n 2^{-(m+1)} \\ &\le \frac{1}{n}\left(\frac{1}{m}-\frac{1}{n}\right) \to 0\end{aligned}$$

So then $\left\|\frac{1}{m}\sum_{j=1}^{m} y_j - \frac{1}{n}\sum_{j=1}^{n} y_j\right\|^2 \to 0$ as $n, m \to \infty$ so it's Cauchy and therefore converges. □

Problem 9. *Show that a linear functional ϕ on a Banach space X is continuous if and only if $\{x \in X \mid \phi(2x) = 3\}$ is norm closed.*

Proof. $\Rightarrow$) $A = \{x \mid \phi(2x) = 3)\} = \{x \mid 2x \in \phi^{-1}(\{3\})\}$. Let $\psi(x) = \phi(2x)$ so $A = \psi^{-1}(\phi^{-1}(\{3\}))$.

$\Leftarrow$) We want to show $\ker(\phi)$ is closed. Note that $\{x \in X \mid \phi(2x) = 3\} = \{x \in X \mid \phi(x) = 3/2\}$.

Pick some $a \in X$ such that $\phi(a) = 3/2$. Then clearly

$$a + \ker(\phi) \subseteq \{x \in X \mid \phi(x) = 3/2\}$$

and if $\phi(x) = 3/2$ then $\phi(x - a) = 0$ so $x = a + (x - a) \in a + \ker(\phi)$.

Thus, $a + \ker(\phi) = \{x \in X \mid \phi(2x) = 3\}$. Therefore $\ker(\phi) = \{x \in X \mid \phi(2x) = 3\} - a$ which is closed. Then

$$\begin{aligned}\phi' : X/\ker(\phi) &\to \mathbb{R}\\ x + \ker(\phi) &\mapsto \phi(x)\end{aligned}$$

is an isomorphism. Let $\pi : X \to X/\ker\phi$ which is also continuous, so $\phi = \phi' \circ \pi$ is continuous.

□

Problem 10. *Let $C^1[0,1]$ be the space of functions $f \in C[0,1]$ such that f' exists and is continuous in $[0,1]$. The space $C^1[0,1]$ is given the supremum norm. Define $T : C^1[0,1] \to C[0,1]$ by $Tf = f'$ for $f \in C^1[0,1]$. Show that T has a closed graph and that T is not bounded. Decide if $C^1[0,1]$ (together with the supremum norm) is a Banach space or not. (Explain your answer).*

Proof. Let $f_n \to f$ and $Tf_n \to f_n' \to g$ in $\|\cdot\|_\infty$.

$$f_n(x) = \int_0^x f_n'(t)dt + f_n(0) \qquad f(x) = \int_0^x f'(t)dt + f(0)$$

Since $f_n \to f$ then $f_n(0) \to f(0)$. Let $G = \int_0^x g(t)dt + f(0)$. Then

$$\|f - G\| \le \|f - f_n\| + \|f_n - G\| \le \|f - f_n\| + \int_0^x \|f_n - g\|_\infty \le \|f - f_n\| + \|f_n - g\|_\infty \to 0.$$

So $f' = g$ meaning T has a closed graph.

To see T is not bounded, consider $f_n = x^n$ so $\|f_n\|_\infty = 1$ but $\|Tf_n\| = \|nx^{n-1}\|_\infty = n \to \infty$.

Thus, by the closed graph theorem, $C^1[0,1]$ is not a Banach space. □

21 January 2014

Problem 1. *Let $(X, \mathcal{M}, \mu)$ be a non atomic measure space with $\mu(X) > 0$. Show that there is a measurable $f : X \to [0, \infty)$, for which*

$$\int f(x) d\mu(x) = \infty.$$

Proof. Take $X = E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots$ such that $\mu(E_1) > \mu(E_2) > \dots > 0$. Define

$$f(x) = \begin{cases} \mu(E_n \backslash E_{n+1})^{-1} & \text{if } x \in E_n \backslash E_{n+1} \\ 0 & \text{if } x \in \bigcap_{n=1}^{\infty} E_n \end{cases}$$

Then $\int f(x) dx = \sum_{n=1}^{\infty} 1 = \infty$. □

Problem 2. *Assume that μ is a finite measure on $\mathbb{R}^n$. Prove that there is a closed set $A \subset \mathbb{R}^n$ with the property that for each closed $B \subsetneq A$ it follows that $\mu(A \backslash B) \neq 0$.*

Proof. Since $\mathbb{R}^n$ is second countable there is some countable basis U_j for $\mathbb{R}^n$. Let

$$A := \mathbb{R}^n \backslash \bigcup \{U_j : \mu(U_j) = 0\}.$$

Of course $\mu(A^c) = 0$ (here we use second countable). Consider a closed subset $B \subset A$. If $\mu(A \backslash B) = 0$, then in fact $\mu(B^c) = 0$. Yet B^c is an open set, so if

$$B^c = \bigcup_{k=1}^{\infty} U_{j_k}$$

for some (U_{j_k}) then $\mu(U_{j_k}) = 0$ for all k. This implies

$$U_{j_k} \subset A^c \, \forall k \Rightarrow B^c \subset A^c \Rightarrow A \subset B \Rightarrow A = B.$$

So if $B \subsetneq A$, $\mu(A \backslash B) \neq 0$ by contrapositive. □

Problem 3. *For a nonnegative function $f \in L_1([0, 1])$, prove that*

$$\lim_{n \to \infty} \int_0^1 \sqrt[n]{f(x)} dx = m(\{x \mid f(x) > 0\}).$$

Proof. Let

$$E_1 = \{x \mid f(x) \geq 1\}$$
$$E_2 = \{x \mid 0 < f(x) < 1\}$$
$$E_3 = \{x \mid f(x) = 0\}$$

Then

$$\int_0^1 f(x)^{1/n}dx = \int_{E_1} f(x)^{1/n}dx + \int_{E_2} f(x)^{1/n}dx + \int_{E_3} f(x)^{1/n}dx$$

For the first integral on E_1, $\lim_n f(x)^{1/n}dx = 1$ and $|f(x)^{1/n}| \leq |f(x)| \in L^1$, so by DCT, $\int_{E_1} f(x)^{1/n}dx = \int_{E_1} dx = m(E_1)$.

For the second integral on E_2, $\lim_n f(x)^{1/n} = 1$ and $|f(x)^{1/n}| \leq 1 \in L^1$ so again by DCT, $\int_{E_2} f(x)^{1/n}dx = \int_{E_2} dx = m(E_2)$. Therefore,

$$\int_0^1 f(x)^{1/n}dx = m(E_1) + m(E_2) = m(\{x \mid f(x) > 0\}).$$

□

Problem 4. *Let f be Lebesgue integrable on $(0,1)$. For $0 < x < 1$ define*

$$g(x) = \int_x^1 t^{-1}f(t)dt.$$

Prove that g is Lebesgue integrable on $(0,1)$ and that

$$\int_0^1 g(x)dx = \int_0^1 f(x)dx.$$

Proof. Notice that

$$\int_0^1 |g(x)|\,dx \leq \int_0^1 \int_x^1 t^{-1}|f(t)|\,dt\,dx \overset{\text{Tonelli}}{=} \int_0^1 \int_0^t t^{-1}|f(t)|dtdx = \int_0^1 |f(t)|dt < \infty$$

since $f \in L^1(0,1)$. So then by Fubini,

$$\int_0^1 g(x)dx = \int_0^1 \int_x^1 t^{-1}f(t)dtdx = \int_0^1 \int_0^t t^{-1}f(t)dxdt = \int_0^1 f(t)dt.$$

□

Problem 5. *Assume that ν and μ are two finite measures on a measurable space $(X, \mathcal{M})$. Prove that*

$$\nu \ll \mu \Leftrightarrow \lim_{n\to\infty} (\nu - n\mu)^+ = 0.$$

Proof. $\Rightarrow$) Note that $\nu - (n+1)\mu < \nu - n\mu$ for all n since μ is positive. Hence if P_n is the positive set for $\nu - n\mu$, $P_{n+1} \subset P_n$, and furthermore the positive set P of $\lim_{n\to\infty}(\nu - n\mu) \subset \bigcap_{n=1}^\infty P_n$ (note that this limit exists as a signed measure, so Hahn decomposition is possible). Since $\lim_{n\to\infty}(\nu -$

$n\mu)(P) \neq 0$, $(\nu - n\mu)(P) \neq 0$ for all n, which implies that $\nu(P) > n\mu(P)$ for all n. But since ν is finite, $\mu(P) = 0$ while $\nu(P) \neq 0$. Hence $\nu \not\ll \mu$.

$\Leftarrow$) Let $\mu(E) = 0$. Then for all $\epsilon > 0$, there exists some N such that for all $n \geq N$,

$$\epsilon > (\nu - n\mu)^+(E) \geq (\nu - n\mu)(E) = \nu(E) - n\mu(E) \geq \nu(E).$$

Letting ϵ approach 0, we have that $\nu(E) = 0$ so that $\nu \ll \mu$. □

Problem 6. *Let (p_n) be a sequence of polynomials which converges uniformly on $[0,1]$ to some function f, and assume that f is not a polynomial. Prove the $\lim_{n\to\infty} \deg(p_n) = \infty$, where $\deg(p)$ denotes the degree of a polynomial p.*

Proof. We proceed by contrapositive. Assume there exists some subsequence (p_{n_k}) such that $\deg(p_{n_k}) \leq M$ for all M. Then $\mathbb{P}_n := \{a_0 + a_1 x + \cdots + a_n x^n : a_i \in \mathbb{C}\}$ is a finite-dimensional vector space and hence is complete. So it is closed, and since $p_{n_k} \to f$, $f \in \mathbb{P}_n$.

Alternative proof. Assume to the contrary and consider the space $\mathcal{P} = \text{span}\{1, x, x^2, \ldots, x^m\}$ with $(p_n) \subseteq \mathcal{P}$. Since $\{1, x, \ldots, x^m\}$ are basis elements and $\mathcal{P}$ is finite dimensional, then any two norms are equivalent on $\mathcal{P}$ and so if $P = \sum_{k=0}^{\infty} a_k x^k$, we can consider the two norms defined by

$$\|P\|_1 := \sup |a_k| \qquad \|P\|_2 := \sup_{x\in[0,1]} |P(x)|$$

Since $\|p_{n_k} - p_{n_\ell}\|_2 \to 0$, then $\|p_{n_k} - p_{n_\ell}\|_1 \to 0$ so $\{a_{n_k}\}$ is Cauchy. Hence, $P = \sum_{\ell=0}^{m} a_\ell x^\ell$ where $a_\ell = \lim_k a_{n_k}^\ell$ is a polynomial of degree at most m and p_n converges uniformly to p. So therefore, $p = f$. Contradiction!

□

Problem 7. *Let (f_n) be sequence of non zero bounded linear functionals on a Banach space X. Show that there is an $x \in X$ so that $f_n(x) \neq 0$, for all $n \in \mathbb{N}$.*

Proof. Let $E_n = \{x \mid f_n(x) = 0\}$ which is closed in X. Assume the result is not true, so for every $x \in X$, there exists some n such that $f_n(x) = 0$ implies $x \in E_n$, that is, $X = \bigcup_n E_n$.

Since X is a Banach space, then by Baire Category Theorem, there exists some n such that $\emptyset \neq \overline{E_n^\circ} = E_n^\circ$.

Thus, there exists some $r > 0$, $x \in X$ such that $B(r, x) \subseteq E_n$.Then for all $y \in X$,

$$r\frac{y}{\|y\|} + x \in x + B(r, 0) = B(r, x)$$

so then if $f_n(r\frac{y}{\|y\|} + x) = 0$, then $\frac{r}{\|y\|} f_n(y) = -f_n(x) = 0$ so $f_n(y) = 0$ so $f_n = 0$.

Thus, by Baire Category, $\bigcup C_n \neq X$. Contradiction! □

Problem 8. *Assume that $T : \ell_1 \to \ell_2$ is bounded, linear and one-to-one. Prove that $T(\ell_1)$ is not closed in ℓ_2.*

Proof. We proceed by contradiction. If $T(\ell^1)$ is closed, then $T(\ell^1)$ is a hilbert space. Since $T : \ell^1 \to T(\ell^1)$ is bijective, by the open mapping theorem. T is open and T^{-1} is bounded so T is an isomorphism. Then $\ell^1 \cong T(\ell^1)$. But ℓ^1 is not reflexive and $T(\ell^1)$ is reflexive, so contradiction.

□

Problem 9. *For a uniformly bounded sequence (f_n) in $C[0,1]$ (i.e. $\sup_{n\in\mathbb{N}} \sup_{\xi\in[0,1]} |f_n(\xi)| < \infty$) show that f_n converges weakly to 0 $\Leftrightarrow \lim_{n\to\infty} f_n(\xi) = 0$ for all $\xi \in [0,1]$.*

Is the equivalence true if we do not assume that (f_n) is uniformly bounded, explain?

Proof. This question is the same as 3 from August 2015.

$\Rightarrow$) $C([0,1])^* = \mathcal{M}[0,1]$ for all $\xi \in [0,1]$, $\delta_\xi \in \mathcal{M}[0,1]$. So $0 = \lim_n \int f_n d\delta_\xi = \lim_n f_n(\xi)$ so then $\lim_n f_n(\xi) = 0$ (note that this does not require uniformly boundedness!)

$\Leftarrow$) Fix $\mu \in \mathcal{M}[0,1]$, we want to show that $\int f_n d\mu \to 0$. Since $|f_n(x)| \leq M$ for all x and all n, then by dominated convergence theorem, $\int f_n d\mu \to 0$.

Finally, consider h_n given by connecting $(0,0), (1/n, n), (2/n, 0)$ and $(1,0)$. So $h_n(\xi) \to$ for all $x \in [0,1]$. But by taking Lebesgue measure, $\int h_n(x) d\mu(x) = 1$ so $f_n \nrightarrow 0$ weakly. □

Problem 10. *Assume that f is measurable and non negative function on $[0,1]^2$ and that $1 \leq r < p < \infty$. Show that*

$$\left(\int_0^1 \left(\int_0^1 f^r(x,y)dy\right)^{p/r} dx\right)^{1/p} \leq \left(\int_0^1 \left(\int_0^1 f^p(x,y)dx\right)^{r/p} dy\right)^{1/r}.$$

Hint: Let $s = p/r$, let $1 < s' < \infty$ be the conjugate of s and let

$$F : [0,1] \to \mathbb{R}_0^+, \quad x \mapsto \int_0^1 f^r(x,y)dy.$$

Then consider for an appropriate function $h \in L_{s'}[0,1]$ the product hF.

Proof. Let $F(x) := \int_0^1 f^r(x,y)dy$. Let $h \in L^{s'}[0,1]$ with $\|h\|_{s'} = 1$ and $h \geq 0$. Then by Tonelli (since $Fh \geq 0$), we have

$$\begin{aligned}\int_0^1 F(x)h(x)dx &= \int_0^1\int_0^1 f^r(x,y)h(x)dydx\\ &= \int_0^1\int_0^1 f^r(x,y)h(x)dxdy\\ &\le \int_0^1 \|f^r(\cdot,y)\|_{p/r}\|h\|_{s'}dy\\ &= \int_0^1\left(\int_0^1 f^p(x,y)dx\right)^{r/p}dy\end{aligned}$$

So then $\int_0^1 F(x)h(x)dx \le \int_0^1\left(\int_0^1 f^p(x,y)dx\right)^{r/p}dy$ for all $\|h\|_{s'}=1$, $h\ge 0$.

Notice that $F\ge 0$ so when $\|h\|_{s'}=1$, we have

$$\|F\|_s = \sup_{\|h\|_{s'}=1}\int_0^1 F(x)h(x)dx = \sup_{\|h\|_{s'}=1,h\ge 0}\int_0^1 F(x)h(x)dx$$

Therefore,

$$\begin{aligned}\sup_{\|h\|_{s'}=1,h\ge0}\int_0^1 F(x)h(x)dx = \|F\|_s = \|\int_0^1 f^r(x,y)\,dy\|_{p/r} &= \left(\int_0^1\left(\int_0^1 f^r(x,y)dy\right)^{p/r}\right)^{r/p}\\ &\le \int_0^1\left(\int_0^1 f^p(x,y)dx\right)^{r/p}dy\end{aligned}$$

So then,

$$\left(\int_0^1\left(\int_0^1 f^r(x,y)dy\right)^{p/r}dx\right)^{1/p} = \left(\int_0^1\left(\int_0^1 f^p(x,y)dx\right)^{r/p}dy\right)^{1/r}.$$

□

22 August 2013

Problem 1. *Let $1\le p\le\infty$ and let $f\in L^p(\mathbb{R})$. For $t\in\mathbb{R}$, let $f_t(x)=f(x-t)$ and consider the mapping $G:\mathbb{R}\to L^p(\mathbb{R})$ given by $G(t)=f_t$. The space $L^p(\mathbb{R})$ is equipped with the usual norm topology.*

(a) Show that G is continuous if $1\le p<\infty$.

Proof. Since $C_c^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $1\le p<\infty$, we can choose $g\in C_c^\infty(\mathbb{R})$ such that

$\|g-f\|_p<\epsilon$. Let $t_n \to t \in \mathbb{R}$. Then $\forall n$,

$$\|f_{t_n}-f_t\|_p \le \|f_{t_n}-g_{t_n}\|_p + \|g_{t_n}-g_t\|_p + \|g_t-f_t\|_p$$

It's easy to see $\|f_{t_n}-g_{t_n}\|_p$ and $\|g_t-f_t\|_p$ are small since $\|g-f\|_p<\epsilon$. For $\|g_{t_n}-g_t\|_p$: let A be a bounded and closed set in $\mathbb{R}$ such that $\bigcup_n \operatorname{supp} g_{t_n} \cup \operatorname{supp} g_t \subset A$ (since $t_n \to t$). Then by a basic real analysis result, $g_{t_n} \to g_t$ uniformly on A. So for sufficiently large n,

$$\|g_{t_n}-g_t\|_p = \left(\int_{\mathbb{R}} |g(x-t_n)-g(x-t)|^p dx\right)^{1/p} \le \left(\int_A \epsilon_0^p\right)^{1/p} = \epsilon\mu(A)^{1/p}.$$

□

(b) Find an f for which the mapping G is not continuous when $p=\infty$ (and justify your answer).

Proof. We will take $f=\chi_{[0,1]}$, so

$$\|\chi_{[0,1]}(t_n)-\chi_{[0,1]}(t)\|_\infty = \|\chi_{[t_n,t_n+1)}-\chi_{[t,t+1)}\|_\infty = 1 \qquad \forall n$$

although $t_n \to t$, we have $\|\cdot\|_\infty \not\to 0$. □

(c) Let $1 \le p,q \le \infty$ be conjugate exponents (i.e. satisfying $\frac{1}{p}+\frac{1}{q}=1$). Let $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ and show that their convolution $h=f*g$ is continuous. Recall

$$h(t)=\int_{-\infty}^{\infty} f(x)g(t-x)dx.$$

Proof. Define $j(x)=g(-x)$ and note that $g(t-x)=j_t(x)$; then we have (by Hölder)

$$|h(t)-h(t_n)| \le \int_{\mathbb{R}} |f(x)||g(t-x)-g(t_n-x)|dx \le \|f\|_p\|g_t-g_{t_n}\|_q$$

This goes to zero when $1<p\le\infty$ (so that $1\le q<\infty$) from part (a).

Also notice that

$$h(t)=\int_{-\infty}^{\infty} f(x)g(t-x)dx = \int_{-\infty}^{\infty} f(t-y)g(y)dy = g*f.$$

So when $p=1$, $q=\infty$ the same is true. □

Problem 2. *(a) For $f \in C_{\mathbb{R}}([0,1])$, show that $f \ge 0$ if and only if $\|\lambda - f\|_u \le \lambda$ for all $\lambda \ge \|f\|_u$, where $\|\cdot\|_u$ denotes the uniform (supremum) norm.*

Proof. $\Rightarrow$) Note that $\lambda 1 - f \ge 0$ whenever $\lambda \ge \|f\|_u$. Hence $\|\lambda - f\|_u = \lambda - \|f\|_u \le \lambda$.

$\Leftarrow$) If there exists some x such that $f(x)<0$, then if $\lambda \ge \|f\|_\infty$ so $\lambda>0$. Then $\|\lambda-f\|_\infty \ge \lambda - f(x) > \lambda$. Contradiction! □

(b) *Suppose $E \subseteq C_{\mathbb{R}}([0,1])$ is a closed subspace containing the constant function 1. For $\phi \in E^*$, we define $\phi \geq 0$ to mean $\phi(f) \geq 0$ whenever $f \in E$ and $f \geq 0$. Show $\phi \geq 0$ if and only if $\|\phi\| = \phi(1)$.*

Proof. $\Rightarrow$) Note $C_{\mathbb{R}}([0,1])$ is a *real* Banach space; that is, $\phi(f) \in \mathbb{R}$ for all $f \in E$, $\phi \in E^*$. We have

$$\|\phi\| = \sup_{\|f\|_u = 1} |\phi(f)| \geq \phi(1).$$

Also for $\|f\|_u = 1$, we have $1 - f \geq 0$ so $\phi(1-f) \geq 0$ implies $\phi(1) \geq \phi(f)$. Moreover, $\phi(1+f) = \phi(1) + \phi(f) \geq 0$ and so $\phi(1) \geq -\phi(f)$ so then $\phi(1) \geq |\phi(f)|$. Therefore, $\phi(1) = \|\phi\|$

$\Leftarrow$) $\phi(1) = \|\phi\| \geq |\phi(f)|$ for all $\|f\|_u \leq 1$. Assume there exists some $f \geq 0$ but $\phi(f) < 0$.

By rescaling we can assume $\|f\|_u < 1$, so then

$$\|\phi\| \geq \phi\left(\frac{1-f}{\|1-f\|}\right) = \frac{1}{\|1-f\|}(\phi(1) - \phi(f)) \geq \phi(1) - \phi(f) > \phi(1) = \|\phi\|$$

which contradicts! □

(c) *If $\phi \in E^*$ and $\phi \geq 0$, show that there is a bounded linear functional ψ on $C_{\mathbb{R}}([0,1])$ so that $\psi \geq 0$ and the restriction of ψ to E is ϕ.*

Proof. By Hahn-Banach, there exists some ψ which is an extension of ϕ such that $\|\psi\| = \|\phi\| = \phi(1) = \psi(1)$. So $\psi \geq 0$ follows from (b). (Note we can choose $\|\psi\| = \|\phi\|$ since $\|\phi\|\|x\|$ is a sublinear functional on $C_{\mathbb{R}}([0,1])$.) □

Problem 3. (a) *Let μ and λ be mutually singular complex measures defined on the same measurable space $(X, \mathcal{M})$ and let $\nu = \mu + \lambda$. Show $|\nu| = |\mu| + |\lambda|$.*

Proof. Let $E \sqcup F = X$ be a Jordan decomposition for μ, λ - say E is λ-null, F is μ-null. Note $\nu \ll |\nu|$; furthermore, if K is measurable such that $|\mu|(K) = |\nu|(K \cap E) + |\nu|(K \cap F) = 0$, then $0 = \nu(K \cap E) = \mu(K \cap E) = \mu(K)$ (and similarly $0 = \lambda(K)$ as well). So $\mu, \lambda \ll |\nu|$.

Write $d\mu = f\, d|\nu|$, $d\lambda = g\, d|\nu|$; then $d\nu = d\mu + d\lambda = f + g\, d|\nu| \Rightarrow d|\nu| = |f+g|\, d|\nu|$. We also see that, if $L \subset E$,

$$0 = \lambda(L) = \int 1_L\, d\lambda = \int_L f\, d|\nu|,$$

so $f|_E \equiv 0$ $|\nu|$-a.e. (similarly $g|_F \equiv 0$ $|\nu|$-a.e.). Therefore $|f+g| = |f| + |g|$ $|\nu|$-a.e., which gives the third equality in the equation

$$|\nu|(K) = \int 1_K\, d|\nu| = \int_K |f+g|\, d|\nu| = \int_K |f| + |g|\, d|\nu| = \int_K d|\mu| + \int_K d|\lambda| = |\mu|(K) + |\lambda|(K).$$

□

(b) *Construct a nonzero, atomless Borel measure on $[0,1]$ that is mutually singular with respect to Lebesgue measure.*

Proof. Let $f \in [0,1]^{[0,1]}$ be the devil's staircase. (I.e., for $c \in C$ - the Cantor set - write $c = \sum_{i=1}^{\infty} a_i 3^{-i}$ where $a_i \in \{0,2\}$. Define $f : C \to [0,1]$ by

$$f(\sum_{i=1}^{\infty} a_i 3^{-i}) = \sum_{i=1}^{\infty} \frac{a_i}{2} 2^{-i},$$

and extend f to $[0,1]$ by setting $f(x) := f(\max\{c \in C : c \leq x\})$.)

We can define a premeasure $\mu_f((a,b]) = f(b) - f(a)$ and use Caratheodory's theorem to get a Lebesgue-Stieltjes measure on $\mathbb{R}$ (which is atomless since f is continuous). Then C, C^c are both Lebesgue sets, $m(C) = 0$, and $\mu_f(C^c) = 0$ (since f is constant on C^c). Since m, μ_f are positive measures (f is increasing), $m \perp \mu_f$. □

Problem 4. *Let $(f_n)_{n=1}^{\infty}$ be a sequence of continuous functions on $[0,1]$ and suppose that for all $x \in [0,1]$, $f_n(x)$ is eventually nonnegative. Show that there is an open interval $I \subseteq [0,1]$ such that for all n large enough, f_n is nonnegative everywhere on I.*

Proof. Let $U_N = \bigcap_{n=N}^{\infty} f_n^{-1}[0,\infty) = \{x \mid f_n(x) \geq 0 \ \forall n \geq N\}$. This is closed. For every $x \in [0,1]$, $f_n(x)$ is eventually non-negative so $[0,1] = \bigcup_N U_N$.

By Baire-Category, there exists some N such that $\emptyset \neq \overline{U_N}^{\circ} = U_N^{\circ}$. So there exists some open $I \subseteq U_N^{\circ} \subseteq [0,1]$ and then for all $n \geq N$, $f_n \geq 0$ on I. □

Problem 5. *Let μ be a nonatomic signed measure on a measure space (X, Ω), with $\mu(X) = 1$. Show that there is a measurable subset $E \subset X$ with $\mu(E) = 1/2$.*

Proof. Note that $\mu^+ < \infty$ since $\mu(X) = 1$. Hence WLOG we may show the result for finite positive measures; in general we can restrict our sets to living inside the set F which is μ^--null (Jordan decomposition).

Also notice that for every $\epsilon > 0$, there exists some $E \subseteq X$ with $0 < \mu(E) < \epsilon$. This is because we can recursively divide our set into two non-trivial sets and choose the smaller one. That is, assume there is some $\varepsilon > 0$ such that $\mu(E) > 0$ implies $\mu(E) \geq \varepsilon$ and consider the following process: set $E_0 = X$. At step i, find $E_i' \subset E_{i-1}$ such that $\mu(E_i') > 0$. So by assumption $\mu(E_i') > \varepsilon$. We have $\min\{\mu(E_i'), \mu(E_i'^c)\} < \mu(E_{i-1})/2$, which has positive measure. Set E_i to be whichever of E_i' or $E_i'^c$ attains this minimum. This process necessarily leads to a contradiction.

Therefore, for all n, we can find a set E_n such that $0 < \mu(E_n) < 2^{-n}$. Let $\mathcal{S} = \{E \subseteq X \mid \mu(E) \leq \frac{1}{2}\}$ ordered by inclusion.

Zorn's Lemma implies that there exists a maximal element E (spell this out!). If $\mu(E) < \frac{1}{2}$ then we can find some $F \subseteq E^c$ with $0 < \mu(F) < \frac{1}{2} - \mu(E)$ but then $\mu(F \cup E) \leq \mu(F) + \mu(E) \leq \frac{1}{2}$ which contradicts maximality. □

Problem 6. *Compute*

$$\lim_{n \to \infty} \int_0^{\infty} \frac{n \sin(x/n)}{x(1+x^2)} dx$$

and justify your computation.

Proof. Let $f_n(x) = \frac{n\sin(x/n)}{x(1+x^2)}$. Recall that $\lim_{t\to 0}\frac{\sin t}{t} = 1$, so $\lim_n \frac{n\sin(x/n)}{x(1+x^2)} = \frac{1}{1+x^2}$ and since $|\sin(x/n)| \le x/n$ for x, n positive,

$$\left|\frac{n\sin(x/n)}{x(1+x^2)}\right| \le \left|\frac{n}{x}\frac{x}{n}\frac{1}{1+x^2}\right| = \frac{1}{1+x^2} \in L^1[0,\infty).$$

Then by DCT,

$$\lim_n \int_0^\infty \frac{n\sin(x/n)}{x(1+x^2)} = \int_0^\infty \lim_n \frac{n\sin(x/n)}{x(1+x^2)} = \arctan(x)|_0^\infty = \frac{\pi}{2}.$$

□

Problem 7. *Prove or disprove: for every real-valued continuous function f on $[0,1]$ such that $f(0) = 0$ and every $\epsilon > 0$, there is a real polynomial p having only odd powers of x, i.e. p is of the form*

$$p(x) = a_1x + a_3x^3 + a_5x^5 + \cdots + a_{2n+1}x^{2n+1},$$

such that $\sup_{x\in[0,1]} |f(x) - p(x)| < \epsilon$.

Proof. Let

$$\mathcal{A} = \{\text{ polynomial with even power}\}$$

so $\mathcal{A}$ is an algebra that separates points. Stone-Weierstrass implies that $\mathcal{A}$ is dense in $C[0,1]$. Note that the collection of polynomials given above can be written as $x\mathcal{A} := \{xa : a \in \mathcal{A}\}$; we may then rephrase the problem statement as asking whether $\overline{x\mathcal{A}} = \{f \in C[0,1] : f(0) = 0\}$.

The problem statement is then **proven** once we make the following claims:

First, if $\mathcal{A} \subset C[0,1]$, $x\overline{\mathcal{A}} \subset \overline{x\mathcal{A}}$. Let $xa \in x\overline{\mathcal{A}}$ be arbitrary and let $a_i \to a$ uniformly for $(a_i) \subset \mathcal{A}$. Then $\|xa_i - xa\|_\infty \le \|x\|_\infty \|a_i - a\|_\infty = \|a_i - a\|_\infty$. So $xa_i \to xa$ and $xa \in \overline{x\mathcal{A}}$.

Next, we claim that $x\overline{\mathcal{A}} = xC[0,1]$ is dense in $\{f \in C[0,1] : f(0) = 0\}$. We quickly see that $xC[0,1]$ is an algebra ($xf + xg = x(f+g)$, $\lambda xf = x(\lambda f)$, and $xfxg = x(xfg)$ where $xfg \in C[0,1]$) that separates points (consider $x1$) but which is non-unital (since $\frac{1}{x} \notin C[0,1]$). Hence Stone-Weierstrass completes the claim.

Since $\overline{x\mathcal{A}}$ is a closed set containing $x\overline{\mathcal{A}}$, the problem statement follows.

□

Problem 8. *Let $f \in L^1_{loc}(\mathbb{R})$.*

(a) *What (by definition) are the Hardy-Littlewood maximal function Hf and the Lebesgue set L_f of f?*

Proof.

$$Hf(x) = \sup_{r>0} \underbrace{\frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y)|dy}_{=:A_r f(x)}.$$

$$L_f = \left\{ x \mid \lim_{r\to 0^+} \frac{\int_{B(r,x)} |f(y)-f(x)|dy}{m(B(r,x))} = 0 \right\}$$

□

(b) *State the Hardy-Littlewood Maximal Theorem.*

Proof. There exists a constant $C > 0$ such that for all $f \in L^1$ and $\alpha > 0$,

$$m(\{x : Hf(x) > \alpha\}) \le \frac{C}{\alpha} \int |f(x)|\, dx.$$

□

(c) *In each case, either construct concretely an example of f with the required property, or explain why no such example exists (you may use theorems from Folland about the Lebesgue set, if you state them).*

(i) $L_f = \mathbb{R}$

(ii) *the complement of L_f is uncountable*

(iii) $L_f \subseteq (-\infty, 0] \cup [1, \infty)$.

Proof. (i) It is easy to see $f \equiv 0$ satisfies the above property, as the integral in the numerator of the conditions for L_f would be 0.

(ii) Consider 1_C, where C is the Cantor set (on $[0,1]$). Since $m(C) = 0$, for $x \in C$ we have

$$\int_{B(r,x)} |f(y)-f(x)|\, dy = \int_{B(r,x)} |f(x)|\, dy = m(B(r,x)),$$

so

$$\lim_{r\to 0^+} \frac{\int_{B(r,x)} |f(y)-f(x)|\, dy}{m(B(r,x))} = 1.$$

Hence L_f^c is an uncountable set.

(iii) By Theorem 3.20, if $f \in L^1_{\text{loc}}$, then $m((L_f)^c) = 0$, so there is no f with this Lebesgue set. □

Problem 9. *Let X be a separable Banach space, let $\{x_n \mid n \ge 1\}$ be a countable, dense subset of the unit ball of X and let B be the closed unit ball in the dual Banach space X^* of X. For $\phi, \psi \in B$, let*

$$d(\phi, \psi) = \sum_{n=1}^{\infty} 2^{-n} |\phi(x_n) - \psi(x_n)|.$$

Show that d is a metric on B whose topology agrees with the weak-topology of X^* restricted to B.*

Proof. We first check that d is a metric on B:

- $d(\phi, \psi) \geq 0$ clear
  If $d(\phi, \psi) = 0$ then $\phi = \psi$ on $\{x_n\}$ so $\phi = \psi$ by continuity / density
- triangle inequality follows as well

the weak*-topology is $\{z^* \in X^* \mid |(x^* - z^*)(x)| < \epsilon\}$ for fixed $x^* \in X^*, x \in X, \epsilon > 0$.

For fixed $x^* \in X^*$ consider the ϵ-ball in the metric d which is

$$\{\psi \in X^* \mid d(x^*, \psi) < \epsilon\} = \{\psi \in X^* \mid \sum_{n=1}^{\infty} 2^{-n}|x^*(x_n) - \psi(x_n)| < \epsilon\}$$

We want to show that $|(x^* - \psi)(x)| < \epsilon'$ for any $x \in B_X$ and some $\epsilon' > 0$. Since x_n is dense in B_X then there exists a $x_{n_k} \to x$. Then

$$|(x^* - \psi)(x)| \leq |(x^* - \psi)(x - x_{n_k})| + |(x^* - \psi)(x_{n_k})| < \epsilon'$$

On the other hand, if ψ is in a weak* neighborhood of x^*, we want to show $\sum_{n=1}^{\infty} 2^{-n}|x^*(x_n) - \psi(x_n)| < \epsilon$. Let $|(\psi - x^*)(x_n)| < \epsilon'$ for all n, then

$$\sum_{n=1}^{\infty} 2^{-n}|x^*(x_n) - \psi(x_n)| < \sum_{n=1}^{\infty} 2^{-n}\epsilon = \epsilon.$$

Alternative Proof. We first check that d is a metric on B:

- $d(\phi, \psi) \geq 0$ clear
  If $d(\phi, \psi) = 0$ then $\phi = \psi$ on $\{x_n\}$ so $\phi = \psi$ by continuity / density
- triangle inequality follows as well

To see that the topologies agree:

Consider $B(r, \varphi)$ under the metric. We need to show it contains an open U under the weak*-topology. Say $d(\phi_k, \psi) \to 0$. Then $\sum_{n=1}^{\infty} 2^{-n}|\phi_k(x_n) - \psi(x_n)| \to 0$. So under the weak* topology, we need to show for all $x \in B_X$, $|\phi_k(x) - \psi(x)| \to 0$.

Indeed, this follows by density of $\{x_n\}$. For large k, $\|\phi_k\| = \sup_n |\phi_k(x_n)| \leq M$ and $|\phi_k(x_n)| \sim |\psi(x_n)|$.

Then for every $\epsilon > 0$, there exists some n such that $\|x_n - x\| < \epsilon$ so

$$|\phi_k(x) - \psi(x)| \leq |\phi_k(x) - \phi_k(x_n)| + |\phi_k(x_n) - \psi(x_n)| + |\psi(x_n) - \psi(x)|$$

If $|\phi_k(x) - \psi(x)| \to 0$ for all x, then for all ϵ, choose N such that $\sum_{n=N}^{\infty} 2^{-n} < \epsilon$, so $d(\phi_k, \psi) = \sum_{n=1}^{\infty} 2^{-n}|\phi_k(x_n) - \psi(x_n)|$. □

Problem 10. *Let $T : X \to Y$ be a linear map between Banach spaces that is surjective and satisfies $\|Tx\| \geq \epsilon\|x\|$ for some $\epsilon > 0$ and all $x \in X$. Show that T is bounded.*

Proof. Is $\Gamma(T) = \{(x, Tx) \mid x \in X\}$ closed in $X \times Y$?

If $x_n \to x$ and $Tx_n \to y$, we want to show $y = Tx$. T is surjective, so $y = Tx_0$. Then for all $\tilde{\epsilon} > 0$, there exists N such that for all $n \geq N$,

$$\tilde{\epsilon} > \|Tx_n - Tx_0\| \geq \epsilon\|x_n - x_0\|.$$

So $x_n \to x_0$, and $Tx_n \to Tx_0$.

The closed graph theorem implies T is bounded. □

23 January 2013

Problem 1. *Let f be a Lebesgue integrable, real-valued function on $(0,1)$ and for $x \in (0,1)$ define*

$$g(x) = \int_x^1 t^{-1} f(t) dt.$$

Show that g is Lebesgue integrable on $(0,1)$ and that $\int_0^1 g(x)dx = \int_0^1 f(x)dx$.

Proof. See January 2014, # 4.

Notice that

$$\int_0^1 |g(x)|\,dx \leq \int_0^1 \int_x^1 t^{-1}|f(t)|\,dt\,dx \overset{\text{Tonelli}}{=} \int_0^1 \int_0^t t^{-1}|f(t)|\,dx\,dt = \int_0^1 |f(t)|\,dt < \infty$$

since $f \in L^1(0,1)$. Note that we have also shown $t^{-1}f(t) \in L^1(m \times m)$ as well. So then by Fubini,

$$\int_0^1 g(x)dx = \int_0^1 \int_x^1 t^{-1}f(t)\,dt\,dx = \int_0^1 \int_0^t t^{-1}f(t)\,dx\,dt = \int_0^1 f(t)\,dt.$$

□

Problem 2. *Let $f_n \in C[0,1]$. Show that $f_n \to 0$ weakly if and only if the sequence $(\|f_n\|)_{n=1}^{\infty}$ is bounded and f_n converges pointwise to 0.*

Proof. See August 2015, # 3.

$\Rightarrow$) We know $C[0,1]^* = \mathcal{M}[0,1]$. Then $f_n \to 0$ weakly implies $\int f_n d\mu \to 0$ for all $\mu \in \mathcal{M}[0,1]$. Choose $\mu = \delta_t$ so

$$\int f_n \, d\delta_t = f_n(t) \to 0 \quad \forall t \in [0,1]$$

(this follows from the fact that weak convergence implies uniformly bounded). Consider

$$\chi : C[0,1] \to C[0,1]^{**} = \mathcal{M}[0,1]^*$$
$$\chi(f_n)(\mu) = \mu(f_n)$$

Since $\mu(f_n) \to 0$ then $\chi(f_n)(\mu) \to 0$ for all $\mu \in \mathcal{M}[0,1]$. Since convergent sequences are bounded, then $\sup_n |\chi(f_n)(\mu)| \le M$. By the uniform boundedness theorem, $\sup_n \|\chi(f_n)\| < \infty$. By isometry, $\|f_n\| = \|\chi(f_n)\|$ so $\sup_n \|f_n\| < \infty$.

$\Leftarrow$) By Dominated Convergence Theorem, $f_n \to 0$ in $L^1(\mu)$ (note that $M1_{[0,1]}$ is a dominating function for all f_n by assumption). So therefore, $|\int f_n d\mu| \le \int |f_n| d|\mu| \to 0$. So $f_n \to 0$ weakly (since μ was arbitrary). ☐

Problem 3. *Let (X, μ) be a measure space with $0 < \mu(X) \le 1$ and let $f : X \to \mathbb{R}$ be measurable. State the definition of $\|f\|_p$ for $p \in [1, \infty]$. Show that $\|f\|_p$ is a monotone increasing function of $p \in [1, \infty)$ and that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.*

Proof. See January 2016, # 8.

By Hölder, we know that $\|f\|_p \le \|f\|_q$ when $p \le q$. (It may be worth going through part of the proof given in Proposition 6.12 for full credit on an exam.) Also, $\|f\|_p \le \|f\|_\infty$ for all p. Therefore, $\|f\|_p \nearrow \|f\|_\infty$ and so $\lim_p \|f\|_p \le \|f\|_\infty$.

On the other hand, for every $\epsilon > 0$, let $E = \{x \mid |f(x)| > \|f\|_\infty - \epsilon\}$ and $0 < \mu(E) \le 1$ since $\|f\|_\infty = \operatorname{esssup} |f(x)|$. Then $\|f\|_p^p \ge \int_E |f|^p > \left(\|f\|_\infty - \epsilon\right)^p \mu(E) \Rightarrow \|f\|_p > (\|f\|_\infty - \varepsilon)\mu(E)^{1/p}$. Take $p \to \infty$ so $\lim_p \|f\|_p \ge \|f\|_\infty - \epsilon$, implying $\lim_p \|f\|_p \ge \|f\|_\infty$. ☐

Problem 4. *(a) Is there a signed Borel measure μ on $[0,1]$ such that*

$$p'(0) = \int_0^1 p(x) d\mu(x)$$

for all real polynomials p of degree at most 19?

Proof. We first define the linear functional $I(p) = p'(0)$.

Write $\mathcal{P} = \operatorname{span}\{1, x, x^2, \dots, x^{19}\}$, which is a finite dimensional space. Thus, all norms are equivalent. We take, in particular, the norms $\|\cdot\|_m = \max_{i=1,\dots,19} |a_i|$ and $\|\cdot\|_\infty$. Then there must exist some C such that if $\|p\|_\infty = 1$ then $\|p\|_m \le C$ so $|a_1| \le C$ which implies that I is bounded.

By Hahn-Banach, there exists some $\tilde{I} \in C[0,1]^*$ such that $\tilde{I}(p) = I(p)$ for all $p \in \mathcal{P}$. By Riesz, there exists some μ such that $\tilde{E}(p) = p'(0) = \int_0^1 p(x) d\mu$. ☐

(b) Is there a signed Borel measure μ on $[0,1]$ such that

$$p'(0) = \int_0^1 p(x)d\mu(x)$$

for all real polynomials p?

Proof. Suppose there did exist such a measure μ on $[0,1]$. Then since $\mu([0,1]) = \int_0^1 1d\mu = 0$, we have that $|\mu|([0,1]) < \infty$. In particular, μ^+ and μ^- are both finite measures, so there is a corresponding bounded linear functional $I \in C[0,1]^*$ such that $I(p) = p'(0)$ for all polynomials p.

But this raises an issue: take the polynomials $p_n := (1-x)^n$. Note $\|p_n\|_\infty = 1$ for all n, but $|p_n'(0)| = n$. So I is unbounded - contradiction. *Thanks Inyoung Ryu for the easy example.* □

Problem 5. *Let $\mathcal{F}$ be the set of all real-valued functions on $[0,1]$ of the form*

$$f(t) = \frac{1}{\Pi_{j=1}^n (t - c_j)}$$

for natural numbers n and for real numbers $c_j \notin [0,1]$. Prove or disprove: for all continuous, real-valued functions g and h on $[0,1]$ such that $g(t) < h(t)$ for all $t \in [0,1]$, there is a function $a \in \operatorname{span} \mathcal{F}$ such that $g(t) < a(t) < h(t)$ for all $t \in [0,1]$.

Proof. Let $\mathcal{A} = \operatorname{span} \mathcal{F}$. It's easy to see this is an algebra since $c_j \notin [0,1]$. Also $\frac{1}{t+1}$ separates points, so Stone-Weierstrass theorem implies $\overline{\mathcal{A}} = C[0,1]$.

Let $M = \min_{t\in[0,1]} |h(t) - g(t)|$, so we can choose some $a \in \mathcal{A}$ such that $\left\| a - \frac{h+g}{2} \right\|_\infty < \frac{M}{6}$. Then $\frac{-M}{6} < a - \frac{h+g}{2} < \frac{M}{6}$ and since $h - g \geq M$, then

$$h - a = \frac{h}{2} - a + \frac{h}{2} \geq \frac{h}{2} - a + \frac{g}{2} + \frac{M}{2} = \frac{h+g}{2} - a + \frac{M}{2} > \frac{M}{2} - \frac{M}{6} = \frac{M}{3} > 0$$

$$a - g = a - \frac{g+g}{2} \geq a - \frac{g+h}{2} + \frac{M}{2} > \frac{-M}{6} + \frac{M}{2} = \frac{M}{3} > 0$$

So then $g < a < h$. □

Problem 6. *Let $k : [0,1] \times [0,1] \to \mathbb{R}$ be continuous and let $1 < p < \infty$. For $f \in L^p[0,1]$, let Tf be the function on $[0,1]$ defined by*

$$(Tf)(x) = \int_0^1 k(x,y)f(y)dy.$$

Show that Tf is a continuous function on $[0,1]$ and that the image under T of the unit ball in $L^p[0,1]$ has compact closure in $C[0,1]$.

Proof. Note that

$$|Tf(x) - Tf(y)| \leq \int_0^1 |k(x,z) - k(y,z)||f(z)|dz \leq \|k(x,\cdot) - k(y,\cdot)\|_q \|f\|_p \qquad \text{for } q = \frac{p}{p-1}$$

Since k is continuous on $[0,1]^2$, then for every $\epsilon > 0$ there exists some $\delta > 0$ such that if $|x-y| < \delta$, then

$$\|k(x,\cdot) - k(y,\cdot)\|_q^q = \int_0^1 |k(x,z) - k(y,z)|^q dz < \int_0^1 \epsilon^p dz = \epsilon^p.$$

Therefore, Tf is continuous.

Now consider $\mathcal{F} = \{Tf \mid \|f\|_p \leq 1\} \subseteq C[0,1]$. We'll use Arzela-Ascoli:

- equicontinuous

  follows from above

- pointwise bounded

$$|Tf(x)| \leq \|K(x,\cdot)\|_q \|f\|_p \leq \|K(x,\cdot)\|_q \leq \left(\int_0^1 M^q dz\right)^{1/q} = M$$

  so it's actually uniformly bounded

Therefore, by Arzela-Ascoli, $\overline{\mathcal{F}}$ is compact in $C[0,1]$. □

Problem 7. *(a) Define the total variation of a function $f : [0,1] \to \mathbb{R}$ and absolute continuity of f.*

Proof. These definitions can be found at (3.24) and (3.31) of Folland. □

(b) Suppose $f : [0,1] \to \mathbb{R}$ is absolutely continuous and defines $g \in C[0,1]$ by

$$g(x) = \int_0^1 f(xy)dy.$$

Show that g is absolutely continuous.

Proof. Since f is absolutely continuous, there exists some $\delta > 0$ such that $\sum_{i=1}^n |b_i - a_i| < \delta$ implies $\sum_{i=1}^n |f(b_i) - f(a_i)| < \epsilon$. Fix some $y \in [0,1]$ so that

$$\sum_{i=1}^n |b_i y - a_i y| \leq \sum_{i=1}^n |b_i - a_i| < \delta'$$

This implies then that $\sum_{i=1}^n |f(b_i y) - f(a_i y)| < \epsilon$. We also note that absolute continuity implies uniform continuity; in particular, f is bounded on $[0,1]$ and is hence in L^1. Therefore,

$$\sum_{i=1}^{n}|g(b_i)-g(a_i)| = \sum_{i=1}^{n}\left|\int_0^1 f(b_iy)-f(a_iy)\,dy\right| \overset{2.25}{\leq} \int_0^1 \sum_{i=1}^{n}|f(b_iy)-f(a_iy)| \leq \int_0^1 \epsilon dx = \epsilon.$$

So g is absolutely continuous. □

Problem 8. *(a) State the definition of absolute continuity, $v \ll \mu$, for positive measures μ and ν, and state the Radon-Nikodym Theorem, (or the Lebesgue-Radon-NIkodym Theorem, if you prefer.)*

Proof. See Theorem 3.8 of Folland. □

(b) Suppose that we have $\nu_1 \ll \mu_1$ and $\nu_2 \ll \mu_2$ for positive measures ν_i and μ_i on measurable spaces $(X_i, \mathcal{M}_i)$ for $i = 1, 2$. Show that we have $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$, and

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x,y) = \frac{d\nu_1}{d\mu_1}(x)\frac{d\nu_2}{d\mu_2}(y).$$

Proof. Assume $E \in \mathcal{M}_1 \otimes \mathcal{M}_2$ and $\mu_1 \times \mu_2(E) = 0$. Define

$$E_x = \{y \in X_2 \mid (x,y) \in E\} \qquad E^y = \{x \in X \mid (x,y) \in E\}$$

Then $E_x \in \mathcal{M}_1$ and $E^y \in \mathcal{M}_2$ for all $x \in X_1, y \in X_2$. Since μ_1 and μ_2 are positive, then $0 = (\mu_1 \times \mu_2)(E) = \int \mu_1(E^y)d\mu_2(y)$ then $\mu_1(E^y) = 0$ μ_2-almsot everywhere and so then $\nu_1(E^y) = 0$ μ_2-almost everywhere.

Thus, $\mu_2(\{y \in X_2 \mid \nu_1(E^y) > 0\}) = 0$ so then $\nu_2(\{y \in X_2 \mid \nu_1(E^y) > 0\}) = 0$. Thus, $\nu_1(E^y) = 0$ for ν_2-almost everywhere and therefore, $(\nu_1 \times \nu_2)(E) = \int \nu_1(E^y)d\nu_2(y) = 0$.

Thus, $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$. By Radon-Nikodym theorem,

$$\nu_1 \times \nu_2(E) = \int_E \frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x,y)d(\mu_1 \times \mu_2) \qquad \text{for } E \in \mathcal{M}_1 \otimes \mathcal{M}_2$$

Since $\nu_1 \ll \mu_1$, by Proposition 3.9(a) in Folland,

$$\begin{aligned}(\nu_1 \times \nu_2)(E) &= \int \nu_2(E_x)d\nu_1(x) \\ &= \int \nu_2(E_x)\frac{d\nu_1}{d\mu_1}(x)d\mu_1(x) \\ &= \int \left(\int_{E_x} \frac{d\nu_2}{d\mu_2}(y)d\mu_2(y)\right)\frac{d\nu_1}{d\mu_1}(x)d\mu_1(x) \\ &= \int_E \frac{d\nu_2}{d\mu_2}(y)\frac{d\nu_1}{d\mu_1}(x)d(\mu_1 \times \mu_2)(x,y)\end{aligned}$$

By the uniqueness of Radon-Nikodym derivative, we have

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x,y) = \frac{d\nu_1}{d\mu_1}(x)\frac{d\nu_2}{d\mu_2}(y).$$

□

Problem 9. *(a) Let E be a nonzero Banach space and show that for every $x \in E$, there is $\phi \in E^*$ such that $\|\phi\| = 1$ and $|\phi(x)| = \|x\|$.*

Proof. See Theorem 5.8(a) in Folland. □

(b) Let E and F be Banach spaces, let $\pi : E \to F$ be a bounded linear map and let $\pi^* : F^* \to E^*$ be the induced map on dual spaces. Show that $\|\pi^*\| = \|\pi\|$.

Proof. We have $\pi^*(y^*)(x) = y^*(\pi(x))$ for all $y^* \in F^*$ and $x \in E$. Then $\|\pi^*(y^*)(x)\| \le \|y^*\|\|\pi\|\|x\|$ so then $\|\pi^*\| \le \|\pi\|$.

On the other hand, by part (a), for each $x \in E$ such that $\|x\| \le 1$, $\pi(x) \in F$, we can find $y^* \in F^*$ such that $|y^*(\pi(x))| = \|\pi(x)\|$ and $\|y^*\| = 1$. Then

$$\|\pi^*\| \ge \|\pi^*(y^*)\| \ge |\pi^*(y)(x)| = |y^*(\pi(x))| = \|\pi(x)\| \qquad \forall \|x\| \le 1$$

So $\|\pi^*\| \ge \|\pi\|$. Thus, $\|\pi\| = \|\pi^*\|$. □

Problem 10. *Let X be a real Banach space and suppose C is a closed subset of X such that*

(i) $x_1 + x_2 \in C$ for all $x_1, x_2 \in C$,

(ii) $\lambda x \in C$ for all $x \in C$ and $\lambda > 0$,

(iii) for all $x \in X$ there exists $x_1, x_2 \in C$ such that $x = x_1 - x_2$.

Prove that, for some $M > 0$, the unit ball of X is contained in the closure of

$$\{x_1 - x_2 \mid x_i \in C, \|x_i\| \le M\}.$$

Deduce that every $x \in X$ can be written $x = x_1 - x_2$, with $x_i \in C$ and $\|x_i\| \le 2M\|x\|$.

Proof. Define

$$C_n = \overline{\{x_1 - x_2 \mid x_i \in C, \|x_i\| \le n\}}$$

By (iii), we know that $X = \bigcup C_n$. By Baire Category, there exists some M such that $\emptyset \ne \overline{C_M}^{\circ} = C_M^{\circ}$. Thus, there exists an open ball $B \subseteq C_M$, $B = B(x_0, 2r)$.

For any $x \in B_X$, $x_0 + rx \in B \subseteq C_M$. From (i), we know that $C_M - C_M \subseteq C_{2M}$ so then $rx = (x_0 + rx) - x_0 \in C_M - C_M \subseteq C_{2M}$. From (ii), we know $x \in C_{2M/r}$, so $B_X \subseteq C_{2M/r}$. Let $M' = \frac{2M}{r}$.

For any $x \in X$, $x \in C_{M'\|x\|}$. So we can find $z_1, y_1 \in C$ such that $\|z_1\|, \|y_1\| \le M\|x\|$ and $\|x - (z_1 - y_1)\| < \frac{1}{2}\|x\|$. Therefore,

$$\frac{2(x-(z_1-y_1))}{\|x\|} \in C_M \quad \Rightarrow \quad x-(z_1-y_1) \in C_{M\|x\|/2}$$

So we can find $z_2, y_2 \in C$ such that $\|z_2\|, \|y_2\| \leq \frac{M}{2}\|x\|$ and

$$\left\| x - \sum_{i=1}^{2}(z_i - y_i) \right\| < \frac{1}{2^2}\|x\|.$$

Inductively, we can find $\{z_n\}, \{y_n\} \subseteq C$ such that $\|z_k\|, \|y_k\| \leq \frac{M}{2^{k-1}}\|x\|$ and

$$\left\| x - \sum_{i=1}^{k}(z_i - y_i) \right\| < \frac{1}{2^k}\|x\|.$$

Then,

$$\sum_{k=1}^{\infty}\|z_k\| \leq \sum_{k=1}^{\infty} M\|x\|\frac{1}{2^k} < 2M\|x\| < \infty$$

so $\sum_{k=1}^{\infty} z_k$ converges to some x_1 in C and similarly $\sum_{k=1}^{\infty} y_k$ converges to some x_2 in C (since C is closed). Moreover,

$$\lim_n \left\| x - \sum_{i=1}^{n}(z_i - y_i) \right\| = \lim_n \left\| x - \left(\sum_{i=1}^{n} z_i - \sum_{i=1}^{n} y_i \right) \right\| = 0.$$

So then $x = \sum_{i=1}^{\infty}(z_i - y_i) = x_1 - x_2$. □

24 August 2012

Problem 1. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Prove that the normed vector space $L^1(X,\mu)$ is complete. You may use any results except the convergence of function series.*

Proof. See class notes. Fill this in! □

Problem 2. *Fix two measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ with $\mu(X), \nu(Y) > 0$. Let $f : X \to \mathbb{C}$, $g : Y \to \mathbb{C}$ be measurable. Suppose $f(x) = g(y)$, $(\mu \times \nu)$-a.e. Show that there is a constant $a \in \mathbb{C}$ such that $f(x) = a$ μ-a.e. and $g(y) = a$ ν-a.e.*

Proof. Let $E := \{(x, y) \in X \times Y \mid f(x) = g(y)\}$, so $(\mu \otimes \nu)(E^c) = 0$. Then for every $a \in \mathbb{C}$, by Fubini-Tonelli,

$$0 = (\mu \otimes \nu)\left(\{(x,y) \in X \times Y \mid f(x) = a, g(y) \neq a\}\right) = \mu\left(\{x \in X \mid f(x) = a\}\right)\nu\left(\{y \in Y \mid g(y) \neq a\}\right).$$

Assume $\mu(\{x \in X \mid f(x) = a\}) = 0$ for all $a \in \mathbb{C}$. Then

$$\begin{aligned} 0 &< (\mu \otimes \nu)(X \times Y) \\ &= (\mu \otimes \nu)(E) = \int_{X \times Y} \chi_{\{(x,y) \mid f(x) = g(y)\}} d\mu(x) d\nu(y) \\ &= \int_Y \left(\int_X \chi_{\{x \mid f(x) = g(y)\}} d\mu(x) \right) d\nu(y) \\ &= \int_Y 0 d\nu(y) = 0. \end{aligned}$$

This is a contradiction so there must exist some $a \in \mathbb{C}$ with $\mu(\{x \in X \mid f(x) = a\}) > 0$. Then $\nu(\{y \in Y \mid g(y) \neq 0\}) = 0$ so $g(y) = a$ ν-a.e.

Similarly, we have $(\mu \otimes \nu)(\{(x,y) \mid f(x) \neq a, g(y) = a\}) = 0$. Since $\nu(\{y \in Y \mid g(y) = a\}) = \nu(Y) \neq 0$, then $\mu(\{x \mid f(x) \neq a\}) = 0$ so $f(x) = a$ μ-a.e. □

Problem 3. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a Borel measurable function. Suppose for every ball B, f is Lebesgue integrable on B and $\int_B f(x)dx = 0$. What can you deduce about f? Justify your answer carefully.*

Proof. Since $f \in L^1_{loc}(\mathbb{R}^2)$, by Lebesgue Differentiation Theorem, for a.e. $x_0 \in \mathbb{R}^2$,

$$\lim_{r \to 0} \frac{1}{|B(r, x_0)|} \int_{B(r, x_0)} f(x) dx = f(x_0)$$

This implies $f(x_0) = 0$ so $f = 0$ almost everywhere. □

Problem 4. *Let X be a locally compact Hausdorff space. Denote by $C_0(X)$ the space of complex-valued continuous functions on X which vanish at infinity, and by $C_c(X)$ the subset of compactly supported functions. Use an approximate version of the Stone-Weierstrass theorem to prove that $C_c(X)$ is dense in $C_0(X)$.*

Proof. For any $f, g \in C_c(X)$, the complex conjugation of f is also in $C_c(X)$.

By complex-LCH-Stone-Weierstrass, we only need to show that $C_c(X)$ separates points.

For every $x \neq y$, we can find open U, V with $x \in U, y \in V$ with $U \cap V = \emptyset$. Since X is LCH, we can require $\overline{U}$ to be compact.

Now $\{x\} \subseteq U \subseteq \overline{U} \subseteq X \backslash V \subseteq X \backslash \{y\}$. Then by Urysohn's Lemma for LCH, we can find a continuous function $f : X \to [0,1]$ such that $f|_{\overline{U}} = 1$ and $f(x) = 0$ outside a compact subset of $X \backslash \{y\}$. So $f(x) = 1$, $f(y) = 0$, and $f \in C_c(X)$.

So $C_c(X)$ separates points. Also, there does not exist any $x_0 \in X$ such that $f(x_0) = 0$ for all $f \in C_c(X)$.

Therefore, by Stone-Weierstrass, $\overline{C_c(X)} = C_0(X)$. $\square$

Problem 5. *Give an example of each of the following. Justify your answers*

(a) *A nowhere dense subset of $\mathbb{R}$ of positive Lebesgue measure*

Proof. Take a fat Cantor set. $\square$

(b) *A closed, convex subset of a Banach space with multiple points of minimal norm.*

Proof. Let $X = L^1[0,1]$, $C = \{f \in X \mid \int_0^1 f(t)dt = 0\}$. It's easy to see that C is closed and convex. The minimum norm of elements in C is 1 because

$$\|f\|_1 = \int_0^1 |f(t)|dt \geq \left|\int_0^1 f(t)dt\right| = 1.$$

But every element of $\{a\chi_{[0,1/2]} + (2-a)\chi_{[1/2,1]}\}_{0\leq a\leq 2}$ in C has norm 1. $\square$

Problem 6. *Let*

$$S = \left\{f \in L^\infty(\mathbb{R}) \mid |f(x)| \leq \frac{1}{1+x^2} \text{ a.e.}\right\}.$$

Which of the following statements are true? Prove your answers.

(a) *The closure of S is compact in the norm topology*

Proof. <u>NO</u>. Let

$$f_n(x) := \frac{1}{x^2+1}\chi_{[\frac{1}{n+1},\frac{1}{n}]}(x)$$

in S. So there are no subsequences of (f_n) which are Cauchy in L^∞ since $\|f_n - f_m\|_\infty \geq 1$ for $n \neq m$. $\square$

(b) *S is closed in the norm topology.*

Proof. <u>YES</u>. Suppose $(f_n) \subseteq S$, $f_n \to f$ in L^∞. Then

$$|f(x)| \leq |f_n(x)| + |f_n(x) - f(x)| < \frac{1}{1+x^2} + \|f_n - f\|_\infty < \frac{1}{1+x^2} + \epsilon \quad \text{a.e.}$$

Letting $\epsilon \to 0$, we have $|f(x)| \leq \frac{1}{1+x^2}$ a.e. and $f \in L^\infty$ so $f \in S$. $\square$

(c) *The closure of S is compact in the weak* topology*

Proof. YES. The unit ball in $L^\infty(\mathbb{R})$ is weak*-compact by Alaoglu. Since $\frac{1}{1+x^2} \leq 1$ for all $x \in \mathbb{R}$, then S is a subset of the unit ball in L^∞. Therefore, $\overline{S}^{w*}$ is weak* compact. □

Problem 7. *Let T be a bounded operator on a Hilbert space $\mathcal{H}$. Prove that $\|T^*T\| = \|T\|^2$. State the results you are using.*

Proof. Clearly, $\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$. On the other hand,

$$\|T\|^2 = \sup_{\|x\|=1} |\langle Tx, Tx\rangle| = \sup_{\|x\|=1} |\langle T^*Tx, x\rangle|.$$

Since for $\|x\| = 1$,

$$|\langle T^*Tx, x\rangle| \leq \|T^*Tx\|\|x\| \leq \|T^*T\|\|x\|^2 \leq \|T^*T\|$$

then $\|T\|^2 \leq \|T^*T\|$. □

Problem 8. *(a) Let g be an integrable function on $[0,1]$. Does there exist a bounded measurable function f such that $\|f\|_\infty \neq 0$ and $\int_0^1 fgdx = \|g\|_1\|f\|_\infty$? Give a construction or a counterexample.*

Proof. YES. For any $g \in L^1$, let $f = \text{sgn}(g)$ where $g(x) \neq 0$, and 1 where $g(x) = 0$. Then $\|f\|_\infty = 1$ and

$$\int_0^1 fg = \int_0^1 |g(x)|dx = \|g\|_1 = \|g\|_1\|f\|_\infty.$$

□

(b) Let g be a bounded measurable function on $[0,1]$. Does there exist an integrable function f such that $\|f\|_1 \neq 0$ and $\int_0^1 fgdx = \|g\|_\infty\|f\|_1$? Give a construction or a counterexample.

Proof. NO. Let $g(x) = x$ on $[0,1]$ so $\|g\|_\infty = 1$, implying $g \in L^\infty[0,1]$. Assume such an $f \in L^1$ exists, so

$$\|f\|_1 = \|f\|_1\|g\|_\infty = \int_0^1 fgdx = \int_0^1 xf(x)dx$$

and also $\|f\|_1 = \int_0^1 |f|dx$ so then $\int_0^1 f(x)xdx = \int_0^1 |f|dx$. Therefore,

$$\int_0^1 |f(x)|dx = \int_0^1 xf(x)dx \leq \int_0^1 x|f(x)|dx \leq \left(1 - \frac{1}{n}\right)\int_0^{1-1/n} |f(x)|dx + \int_{1-1/n}^1 |f(x)|dx$$

So then

$$\int_0^{1-1/n} |f(x)|dx + \int_{1-1/n}^1 |f(x)|dx \leq \left(1 - \frac{1}{n}\right)\int_0^{1-1/n} |f(x)|dx + \int_{1-1/n}^1 |f(x)|dx$$

Thus, $\int_0^{1-1/n} |f(x)|dx = 0$ for all $n \in \mathbb{N}$. Letting $f_n(x) = \chi_{[0,1-1/n]}|f(x)| \nearrow |f(x)|$ then by monotone convergence theorem, $\int |f(x)dx = \lim_n \int f_n(x) = 0$ so $\|f\|_1 = 0$. □

Problem 9. *Let $F : \mathbb{R} \to \mathbb{C}$ be a bounded continuous function, μ the Lebesgue measure, and $f, g \in L^1(\mu)$. Let*

$$\tilde{f}(x) = \int F(xy)f(y)d\mu(y), \qquad \tilde{g}(x) = \int F(xy)g(y)d\mu(y).$$

Show that $\tilde{f}$ and $\tilde{g}$ are bounded continuous functions which satisfy

$$\int f\tilde{g}d\mu = \int \tilde{f}gd\mu.$$

Proof. We have $\|\tilde{f}\|_\infty \leq \|F\|_\infty\|f\|_1 < \infty$ and $\|\tilde{g}\| \leq \|F\|_\infty\|g\|_1 < \infty$ so $\tilde{f}, \tilde{g} \in L^\infty$. By dominated convergence theorem, we know that $\lim_n \int_{[-n,n]} |f(x)d\mu = \|f\|_1$. So then for every $\epsilon > 0$, there exists some N such that $\int_{\mathbb{R}\setminus[-n,n]} |f(x)|d\mu < \epsilon$. Then

$$\begin{aligned}
|\tilde{f}(x_1) - \tilde{f}(x_2)| &\leq \int |F(x_1y) - F(x_2y)||f(y)|d\mu(y) \\
&= \int_{[-n,n]} |F(x_1y) - F(x_2y)||f(y)|d\mu(y) + \int_{\mathbb{R}\setminus[-n,n]} |F(x_1y) - F(x_2y)||f(y)|d\mu(y) \\
&\leq \sup_{y\in[-n,n]} |F(x_1y) - F(x_2y)|\|f\|_1 + 2\|F\|_\infty\epsilon
\end{aligned}$$

Since F is continuous, let $|x_1 - x_2| < \frac{\delta}{n}$ such that $|x_1y - x_2y| < \delta$ imples $|F(x_1y) - F(x_2y)| < \epsilon$. So $|\tilde{f}(x_1) - \tilde{f}(x_2)| \to 0$ as $|x_1 - x_2| \to 0$.

A similar argument will show that $\tilde{g}$ is continuous. Since $f\tilde{g} \in L^1$, by Fubini,

$$\begin{aligned}
\int f\tilde{g}d\mu &= \int\int f(x)F(xy)g(y)d\mu(y)d\mu(x) \\
&= \int g(y)\left(\int f(x)F(xy)d\mu(x)\right)d\mu(y) \\
&= \int g(y)\tilde{f}(y)d\mu(y) \\
&= \int \tilde{f}gd\mu.
\end{aligned}$$

□

Problem 10. *Let μ, $\{\mu_n \mid n \in \mathbb{N}\}$ be finite Borel measures on $[0,1]$. $\mu_n \to \mu$ vaguely if it converges in the weak* topology on $M[0,1] = (C[0,1])^*$. $\mu_n \to \mu$ in moments if for each $k \in \{0\} \cup \mathbb{N}$,*

$$\int_{[0,1]} x^k d\mu_n(x) \to \int_{[0,1]} x^k d\mu(x).$$

Show that $\mu_n \to \mu$ vaguely if and only if $\mu_n \to \mu$ in moments.

Proof. $\Rightarrow$) trivial by the definitions

$\Leftarrow$) We want to show that for all $f \in C[0,1]$, $\int f d\mu_n \to \int f d\mu$. By Stone-Weierstrass, we can find p_n to be a sequence of polynomials which converge uniformly to f on $[0,1]$

$$\left|\int f(x)d\mu - \int f(x)d\mu_n\right| \leq \left|\int f d\mu - \int p_m d\mu\right| + \left|\int p_m d\mu - \int p_m d\mu_n\right| + \left|\int p_m d\mu_n - \int f d\mu_n\right|$$

For the first part, $|\int f d\mu - \int p_m d\mu| \leq \|f - p_m\|_\infty \mu(X) \to 0$ as $m \to \infty$. Similarly, $\left|\int p_m d\mu_n - \int f d\mu_n\right| \leq \|f - p_m\|_\infty \mu_n(X) \to 0$ for all n.

Next, find a polynomial q_{m_j} with degree at most j such that $\|q_{m_j} - p_m\|_\infty \to 0$ as $j \to \infty$. Then since $\mu_n \to \mu$ in moments, then $\left|\int q_{m_j} d\mu - \int q_{m_j} d\mu_n\right| \to 0$ for all j. Thus,

$$\begin{aligned}\left|\int p_m d\mu - \int p_m d\mu_n\right| &\leq \left|\int p_n d\mu - \int q_{m_j} d\mu\right| + \left|\int q_{m_j} d\mu - \int q_{m_j} d\mu_n\right| + \left|\int q_{m_j} d\mu_n - \int p_m d\mu_n\right| \\ &\leq \|p_m - q_{m_j}\|_\infty \left(\mu(X) + \mu_n(X)\right) + \left|\int q_{m_j} d\mu - \int q_{m_j} d\mu_n\right| \to 0.\end{aligned}$$

□

25 January 2012

Problem 1. *Let $\mathcal{A}$ be the subset of $[0,1]$ consisting of numbers whose decimal expansions contain no sevens. Show that $\mathcal{A}$ is Lebesgue measurable, and find its measure. Why does non-uniqueness of decimal expansions not cause any problems?*

Proof. Let A_i be the subset of $[0,1]$ consisting of numbers whose first i digits are not 7. Then $A_{n+1} \subseteq A_n$ and $A = \bigcap_n A_n$,

$$\begin{aligned} A_1 &= [0, 0.7] \cup [0.8, 1] \\ A_2 &= [0, 0.07] \cup [0.08, 0.17] \cup \ldots \cup [0.98, 1] \end{aligned}$$

So A_n is the union of some Borel intervals in $[0,1]$, so A_n is Lebesgue measurable. Therefore, A is Lebesgue measurable.

Now for $0 \leq i \leq 9$, let A_n^i be the subset of A_n such that the $(n+1)$th digit is i. Then we can write $A_n = \sqcup_{i=0}^{9} A_n^i$.

Also, $m(A_n^i) = m(A_n^j)$, so $m(A_n) = 10m(A_n^i)$ and $A_{n+1} = \sqcup_{i \neq 7} A_n^i$ so $m(A_{n+1}) = 9m(A_n^i)$. Therefore, $m(A_{n+1}) = \frac{9}{10} m(A_n)$. Then

$$m(A) = m\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_n m(A_n) = \lim_n \left(\frac{9}{10}\right)^{n-1} m(A_1) = 0$$

The only numbers with non-unique decimal representation are $0.a_1a_2\ldots a_n = 0.a_1a_2\ldots a_{n-1}999\ldots$. However $\forall n$ there are only finitely many, so non-unique $= \bigcup_n \{0.a_1\ldots a_n\}$ which is countable, hence null, hence Lebesgue. $\square$

Problem 2. *Let the functions f_α be defined by*

$$f_\alpha(x) = \begin{cases} x^\alpha \cos(1/x) & x > 0 \\ 0 & x = 0 \end{cases}$$

Find all values of $\alpha \geq 0$ such that

(a) f_α is continuous

Proof. When $a > 0$, $x^a \cos(1/x) \leq x^a \to 0$ as $x \to 0$ so f_a is continuous. If $a = 0$, we know $\cos(1/x)$ isn't continuous at 0. $\square$

(b) f_α is of bounded variation on $[0, 1]$

Proof. First, $0 < a \leq 1$, put partitions

$$P_m = \left\{0, \frac{1}{2\pi m}, \frac{1}{\pi(2m-1)}, \ldots, \frac{1}{\pi}, 1\right\}$$

Then

$$f_a(P_m) = \left\{0, \frac{1}{(\pi 2m)^a}, \frac{-1}{(\pi(2n-1))^a}, \ldots, \frac{-1}{\pi^a}, \cos(1)\right\}$$

So

$$T_{f_\alpha}(P_m) = \left|\frac{1}{(\pi(2m))^a} - 0\right| + \left|\frac{-1}{(\pi(2m-1))^a} - \frac{1}{(\pi 2m)^a}\right| + \ldots + \left|\cos(1) - \frac{-1}{\pi^a}\right| \approx \sum_{i=1}^{2m} \frac{c}{(\pi i)^a} \to \infty$$

when $0 < a \leq 1$ as $m \to \infty$.

So when $0 < a \leq 1$, f_a is not of bounded variation when $0 < a \leq 1$. For $a > 1$, let's look at (c). $\square$

(c) f_α *is absolutely continuous on* $[0,1]$

Proof. When $a > 1$, we see $f_a'(0) = 0$ and f_a' is integrable because $f_a'(x) = ax^{a-1}\cos(1/x) + x^{a-2}\sin(1/x)$ so then $f_a(x) = \int_0^x f_a'(t)dt$. Thus, f is absolutely continuous.

So in (b) we have f_a is of bounded variation for $a > 1$. Since f_a isn't bounded variation when $0 < a \le 1$, so f_a isn't absolutely continuous either when $0 < a \le 1$. □

Problem 3. *Let* $\mathcal{F}$ *denote the family of functions on* $[0,1]$ *of the form*

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$$

where a_n *are real and* $|a_n| \le 1/n^3$. *State a general theorem and use that theorem to prove that any sequences in* $\mathcal{F}$ *has a subsequence that converges uniformly on* $[0,1]$.

Proof. We'll use Arzela-Ascoli II.

For all $f \in \mathcal{F}$,

$$|f(x)| = \left|\sum_{n=1}^{\infty} a_n \sin(nx)\right| \le \sum_{n=1}^{\infty} |a_n| \le \sum_{n=1}^{\infty} n^{-3} < \infty$$

so uniformly bounded. Also, for all $f \in \mathcal{F}$,

$$|f(x)-f(y)| \le \sum_{n=1}^{\infty} |a_n||\sin(nx)-\sin(ny)| \le \sum_{n=1}^{\infty} 2n^{-3}\left|\cos\frac{nx+ny}{2}\right|\left|\sin\frac{nx+ny}{2}\right| \le \sum_{n=1}^{\infty} n^{-2}|x-y| = \frac{\pi^2}{6}|x-y|$$

So $\mathcal{F}$ is equicontinuous.

Then the result follows by A-A II. (A-A I is sufficient since $C[0,1]$ is a metric space.) □

Problem 4. *Let* $\mathcal{H}$ *be a Hilbert space and* $W \subset \mathcal{H}$ *a subspace. Show that* $\mathcal{H} = \overline{W} \oplus W^\perp$ *where* $\overline{W}$ *is the closure of* W.

Note: Do not just state this as a consequence of a standard result, prove the result.

Proof. The "standard result" being referred to here is Theorem 5.24 in Folland. You may need to show that $W^\perp = \overline{W}^\perp$. Happy reading :) □

Problem 5. *Suppose* A *is a bounded linear operator on a Hilbert space* $\mathcal{H}$ *with the property that*

$$\|p(A)\| \le C \sup\{|p(z)| \mid z \in \mathbb{C}, |z| = 1\}$$

for all polynomials p *with complex coefficients, and a fixed constant* C. *Show that to each pair* $x, y \in \mathcal{H}$ *there corresponds a complex Borel measure* μ *on the circle* $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ *such that*

$$\langle A^n x, y\rangle = \int z^n d\mu(z) \qquad n = 0, 1, 2, \ldots$$

Proof. Consider

$$T_{x,y} : P(S^1) \to \mathbb{C}$$
$$p \mapsto \langle P(A)x, y\rangle$$

Then

$$|\langle P(A)x, y\rangle| \le \|P(A)\|\|x\|\|y\| \le C\|P\|_\infty \|x\|\|y\|$$

Thus, $|T_{x,y}(P)| \le C\|x\|\|y\|\|P\|_\infty = f(P)$ which is obviously a seminorm. By Hahn-Banach, $T_{x,y}$ can be extended to $C(S^1)$.

Then apply Riesz-Representation Theorem, there exists a complex Borel measure μ on S^1 such that

$$T_{x,y}(P) = \langle P(A)x, y\rangle = \int_{S^1} P(z) d\mu(z)$$

Take $P(z) = z^n$ so $\langle A^n x, y\rangle = \int_{S^1} z^n d\mu(z)$. □

Problem 6. *Let ϕ be the linear functional*

$$\phi(f) = f(0) - \int_{-1}^{1} f(t)dt$$

(a) *Compute the norm of ϕ as a functional on the Banach space $C[-1, 1]$ with uniform norm*

Proof.

$$|\phi(f)| \le |f(0)| + \int_{-1}^{1} |f(t)|dt \le \|f\|_\infty + \|f\|_\infty \int_{-1}^{1} dt = 3\|f\|_\infty$$

So $\|\phi\| \le 3$. On the other hand, let f_n be piecewise linear functional such that $f_n = -1$ on $[-1, -1/n]$ and $[1/n, 1]$ and $f_n(0) = 1$. Then

$$\int_{-1}^{1} f_n(t)dt = -2(1 - 1/n) + \frac{2}{n} = -2 + \frac{4}{n} \to -2$$

So $\sup |\phi(f_n)| \ge 3$ so $\|\phi\| = 3$. □

(b) *Compute the norm of ϕ as a functional on the normed vector space $LC[-1, 1]$ which is $C[-1, 1]$ with the L^1 norm.*

Proof.

$$\|\phi\| = \sup_{f \in LC[-1,1]} \frac{|f(0) - \int_{-1}^{1} f(t)dt|}{\|f\|_1} \ge \lim_n \frac{|1 - 1/n|}{(1/n)} = \infty$$

□

Problem 7. *Let X be a normed space and $\mathcal{A} \subset X$ be a subset. Show that $\mathcal{A}$ is bounded (as a set) if and only if it is weakly bounded (that is, $f(A) \subset \mathbb{C}$ is bounded for each $f \in X^*$).*

Proof. $\Rightarrow$) for all $x \in A$, for all $f \in X^*$, $|f(x)| \leq \|f\|\|x\| < \infty$ so A is weakly bounded

$\Leftarrow$) on the other hand, consider $A^{**} = \{a^{**} \mid a \in A\}$ by $a^{**}(f) = f(a)$ for all $f \in X^*$. Since X^* is Banach, and we know

$$\sup_{a^{**} \in A^{**}} \|a^{**}(f)\| = \sup_{a \in A} |f(a)| < \infty \qquad \forall f \in X^*$$

By the uniform boundedness principle, $\sup_{a \in A} \|a\| = \sup_{a^{**} \in A^{**}} \|a^{**}\| < \infty$. □

Problem 8. *Let X be a topological vector space.*

(a) Define what this means.

Proof. Let X be a vector space, $\mathcal{T}$ a topology on X. Then $(X, \mathcal{T})$ is a topological vector space provided

- $+ : X \times X \to X$ is continuous
- $\cdot : \mathbb{R} \times X \to X$ is continuous

□

(b) Let $A \subset X$ be compact and $B \subset X$ be closed. Show that $A + B \subset X$ is closed.

Proof. Fix $z \in (A+B)^c$. For $x \in A$, $z - x \in B^c$ so there exists an open neighborhood $V_x \ni 0$ in X such that $(z - x + V_x) \cap B = \emptyset$. Since addition is continuous, there exists U_{1x}, U_{2x} neighborhoods of 0 such that $U_{1x} + U_{2x} \subseteq V_x$.

$U_x = U_{1x} \cap U_{2x} \cap (-U_{1x}) \cap (-U_{2x})$ so $U_x = -U_x$. Then $\{x + U_x\}_{x \in A}$ is an open cover of A. Since A is compact, there exists a finite subcover $x_1, \ldots, x_n \in A$ such that

$$A \subseteq \bigcup_{i=1}^{n} x_i + U_{x_i}$$

Put $U = \bigcap_{i=1}^{n} U_{x_i}$. Then $z + U$ is an open neighborhood of z. If there exists $x \in A$ $y \in B$ such that $x + y \in z + U$ then $x \in x_i + U_{x_i}$ for some i and $y \in z - x + U \subseteq z - x_i + U_{x_i} \subseteq z - x_i + V_{x_i}$ but $(z - x_i + V_{x_i}) \cap B = \emptyset$. Contradiction!

So $(z + U) \cap (A + B) = \emptyset$. □

(c) Give an example indicating that the condition 'A closed' is insufficient for the conclusion.

Proof. $X = \mathbb{R}^2$, $A = \{(x, 0) \mid x \in \mathbb{R}\}$ and $B = \{(x, 1/x) \mid x > 0\}$. Then $A + B = \{(x, y) \mid y > 0\}$. □

Problem 9. *Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Let $f, f_n \in L^3(X, d\mu)$ for $n \in \mathbb{N}$ be functions such that $f_n \to f$ μ-a.e. and $|f_n| \le M$ for all n. Let $g \in L^{3/2}(X, d\mu)$. Show that*

$$\lim_n \int f_n g d\mu = \int f g d\mu.$$

Proof. $|f_n g| \le M|g|$. Since μ is a finite measure, $M1 \in L^3(\mu)$. By Holder, $M|g| \in L^1(\mu)$. The result follows from Dominated Convergence Theorem. □

Problem 10. *Let X be a σ-finite measure space, and $f_n : X \to \mathbb{R}$ a sequence of measurable functions on it. Suppose $f_n \to 0$ in L^2 and L^4.*

(a) Does $f_n \to 0$ in L^1?

Proof. NOT NECESSARILY.

Let $X = \mathbb{R}$, μ=Lebesgue measure. $f_n = n^{-1}\chi_{[0,n]}$ so $\|f_n\|_1 = 1$ does not converge to 0, but $\|f_n\|_2 = n^{-1/2} \to 0$ and $\|f_n\|_4 = n^{-3/4} \to 0$. □

(b) Does $f_n \to 0$ in L^3?

Proof. YES.

Since $0 < 2 < 3 < 4 < \infty$, $L_2 \cap L_4 \subseteq L_3$ and $\|f\|_3 \le \|f\|_2^{\lambda}\|f\|_4^{1-\lambda}$ where $\frac{1}{3} = \frac{\lambda}{2} + \frac{1-\lambda}{4}$ implies $\lambda = \frac{1}{3}$. So

$$\|f_n\|_3 \le \|f_n\|_2^{1/3}\|f_n\|_4^{2/3} \to 0$$

□

(c) Does $f_n \to 0$ in L^5?

Proof. NOT NECESSARILY.

$X = [0, 1]$, μ: Lebesgue measure. Let $f_n = n\chi_{[0,n^{-5}]}$. Then $\|f_n\|_5 = 1$ but $\|f_n\|_2 = n^{-3/2} \to 0$ and $\|f_n\|_4 = n^{-1/4} \to 0$. □

26 August 2011

Problem 1. *Let $(X, \mathcal{M}, \mu)$ be a measure space.*

(a) Give the definitions of convergence a.e. and convergence in measure for a sequence of measurable functions on X.

Proof. We say a sequence of measurable functions f_n converge to f almost everywhere if $\mu(\{x \mid \lim_n f_n(x) \ne f(x)\}) = 0$.

We say that f_n converges to f in measure if $\forall \epsilon > 0$, $\lim_n \mu(\{x \mid |f(x) - f_n(x)| > \epsilon\}) = 0$. □

(b) *Show that every sequence of measurable functions on X which converges in measure to 0 has a subsequence which converges a.e. to 0.*

Proof. Suppose for every $\epsilon > 0$, $\mu(\{x \mid |f_n(x)| \geq \epsilon\}) \to 0$. Choose a subsequence $\{f_{n_k}\}$ such that if

$$E_j = \{x \mid |f_{n_j}(x) - f_{n_{j+1}}(x)| > 2^{-j}\}$$

satisfies $\mu(E_j) < 2^{-j}$. Let $F_k = \bigcup_{j=k}^{\infty} E_j$ so $\mu(F_k) \leq \sum_{j=k}^{\infty} 2^{-j} \leq 2^{1-k}$. Let $F = \bigcap_k F_k$ so $\mu(F) = 0$.

For $x \notin F_k$ and for $i \geq j \geq k$ then

$$|f_{n_i}(x) - f_{n_j}(x)| \leq \sum_{\ell=j}^{i-1} |f_{n_\ell}(x) - f_{n_{\ell+1}}(x)| \leq \sum_{\ell=j}^{i-1} 2^{\ell} \leq 2^{-j} \to 0 \quad \text{as } k \to \infty.$$

So f_{n_k} is pointwise Cauchy on $x \notin F$, so let

$$f(x) = \begin{cases} \lim f_{n_k}(x) & x \notin F \\ 0 & \text{otherwise} \end{cases}$$

So $f_{n_k} \to 0$ almost everywhere and $f_n \to f$ in measure since

$$\mu(\{x \mid |f_n(x) - f(x)| \geq \epsilon\}) \leq \underbrace{\mu(\{x \mid |f_n(x) - f_{n_\ell}(x)| \geq \epsilon/2\})}_{\to 0} + \underbrace{\mu(\{x \mid |f_{n_\ell}(x) - f(x)| \geq \epsilon\})}_{\to 0}$$

and

$$\mu(\{x \mid |f(x)| \geq \epsilon\}) \leq \underbrace{\mu(\{x \mid |f(x) - f_n(x)| \geq \epsilon/2\}}_{\to 0} + \underbrace{\mu(\{x \mid |f_n(x)| \geq \epsilon/2\})}_{\to 0}$$

so $f = 0$ almost everywhere. Thus, $\{f_{n_k}\}$ converges to 0 almost everywhere. □

Problem 2. *Let X be a separable Banach space. Show that there exists an isometric linear map from X into ℓ^∞. Also, show that this is false in general if ℓ^∞ is replaced by ℓ^2.*

Proof. Let (x_n) be a dense sequence in B_X. For each n, use Hahn-Banach Theorem to find a norm-one functional $f_n \in X^*$ with $f_n(x_n) = 1$.

Define $\phi : X \to \ell^\infty$ via $\phi(x) = \big(f_n(x)\big)$. Suppose $x \in X$ has norm one and let $1 > \epsilon > 0$. Choose n_ϵ so that $\|x_{n_\epsilon} - x\| < \epsilon$. Then

$$\epsilon > |f_{n_\epsilon}(x_{n_\epsilon} - x)| = |f_{n_\epsilon}(x)|$$

So $\|\phi(x)\| = \sup_n |f_n(x)| \geq 1$. For every n, $|f_n(x)| \leq \|f_n\|\|x\| = 1$ so $\|\phi(x)\| \leq 1$. So $\|\phi(x)\| = 1$ whenever $\|x\| = 1$. Then for all non-zero x, $\|\phi(x)\| = \|x\| \sup_n |f_n(x/\|x\|)| = \|x\|$. So ϕ is an isometry.

This is false in general if ϕ were to have its range in ℓ^2. Define a map $\langle \cdot, \cdot \rangle_X : X \to \mathbb{C}$ to be

$$\langle x, y \rangle_X := \langle \phi(x), \phi(y) \rangle_{\ell^2}.$$

Since ϕ is linear and $\langle \cdot, \cdot \rangle_{\ell^2}$ is sesquilinear, $\langle \cdot, \cdot \rangle_X$ is sesquilinear. Also, $\langle x, x \rangle_X = \langle \phi(x), \phi(x) \rangle_{\ell^2} > 0$ since ϕ is an isometry $x \neq 0 \Rightarrow \phi(x) \neq 0$. Hence $\langle \cdot, \cdot \rangle_X$ is an inner product, and since

$$\|x\|^2 = \|\phi(x)\|^2 = \langle \phi(x), \phi(x) \rangle = \langle x, x \rangle,$$

so this inner product generates the norm on X. Since X is a Banach space, it is complete with respect to the inner product, so X is a Hilbert space. But not all separable Banach spaces are Hilbert spaces: for example, $\ell^p[0, 1]$ for $1 \leq p < \infty$, $p \neq 2$. □

Problem 3. *Let X be a locally compact metric space and let $\{x_k\}$ be a sequence in X which has no convergent subsequence. Show that $\{n^{-1} \sum_{k=1}^n \delta_{x_k}\}$ converges to 0 in the weak* topology on $C_0(X)^*$, where δ_{x_k} denotes the point mass at x_k.*

Proof. Recall that $\delta_x(f) = f(x)$, so $\delta_x \in C_0^(X)^*$. So we want to show $\frac{1}{n} \sum_{k=1}^n \delta_{x_k}(f) = \frac{1}{n} \sum_{n=1}^\infty f(x_k) \to 0$ in $\mathbb{C}$. Since (x_k) has no convergent subsequence, by definition of compact (x_k) is not frequently in any set in the collection

$$A_m := \{x : |f(x)| \geq \frac{1}{m}\}_{m=1}^\infty.$$

Let N_m be the number such that $n > N_m \Rightarrow x_k \notin A_m$. Now $|f|$ is bounded by some number M, so for $n \gg N_m$ $\frac{1}{n} \sum_{k=1}^n f(x_k) < \frac{2}{m}$. Letting $m \to \infty$, we are done. □

Problem 4. *Let $\mathcal{P}$ be the set of all polynomials f on $[0, 1]$ such that $f(0) = f'(0) = 0$. Determine, with proof, the values of p with $1 \leq p \leq \infty$ such that $\mathcal{P}$ is dense in $L^p[0, 1]$.*

Proof. All $1 \leq p < \infty$. Clearly, $\mathcal{P}$ is an algebra which separates points (ex. x^2). Stone-Weierstrass implies $\overline{\mathcal{P}} = \{f \in C[0, 1] \mid f(0) = 0\}$. Now for any $f \in L^p$, for all $\epsilon > 0$, there exists some N such that

$$\left\| f - f\chi_{[-N \leq f \leq N]} \right\|_p \leq \frac{\epsilon}{2}$$

Define $f_N = f\chi_{[-N \leq f \leq N]}$. By Lusin's theorem, there exists a closed set F such that $m([0, 1] \backslash F) \leq \frac{1}{2^p} \frac{\epsilon^p}{(2N)^p} = \frac{1}{2^p} \left(\frac{\epsilon}{2N}\right)^p$. and $f_N|_F$ continuous.

Tietze extension theorem applied to f_N and F implies the extension g is still bounded by N. Then

$$\|f_N - g\|_p^p = \int_{[0,1] \backslash F} |f_N - g|^p \leq (2N)^p m([0, 1] \backslash F) \leq \frac{\epsilon^p}{2^p}$$

So then

$$\|f - g\|_p \leq \|f - f_N\|_p + \|f_N - g\|_p \leq \frac{\epsilon^p}{2^p} + \frac{\epsilon}{2} \leq \epsilon$$

NOT $L^\infty[0,1]$ since $\overline{\mathcal{P}} = \{f \in C[0,1] \mid f(0) = 0\}$. If $f \in L^\infty[0,1]$ with $f(0) = a \neq 0$, then $\forall g \in \overline{\mathcal{P}}$, $\|f - g\|_\infty = a$. □

Problem 5. *Let $1 < p < \infty$ and let $\{x_k\}_{k=1}^\infty$ be a sequence in $\ell^p(\mathbb{N})$ such that $\lim_k x_k(n) = 0$ for all $n \in \mathbb{N}$. Show that if there is an $M > 0$ such that $\|x_k\| \leq M$ for all $k \in \mathbb{N}$ then $x_k \to 0$ weakly.*

Also, show that if no such M exists, then $\{x_k\}$ can fail to converge weakly.

Proof. Note: Similar to August 2015, #3, just in a different space now.

Fix some $y \in \ell^q$ where $\frac{1}{p} + \frac{1}{q} = 1$. We want to show that $\sum_n x_k(n)y(n) \to 0$ as $k \to \infty$. Fix $\epsilon > 0$.

Then we may choose a finite $A \subseteq \mathbb{N}$ such that $\sum_{A^c} |y(n)|^q < \epsilon^q$. Since A is finite, choose some K such that for all $k \geq K$ we have $|x_k(n)|^p < \frac{\epsilon^p}{|A|}$. Then for all $k \geq K$, by using Holder, we have

$$\begin{aligned}
|y(x_k)| &\leq \sum_{n\in\mathbb{N}} |x_k(n)||y(n)| \\
&= \sum_{n\in A} |x_k(n)||y(n)| + \sum_{n\in A^c} |x_k(n)||y(n)| \\
&\leq \left(\sum_{n\in A} |x_k(n)|^p\right)^{1/p} \left(\sum_{n\in A} |y(n)|^q\right)^{1/q} + \left(\sum_{n\in A^c} |x_k(n)|^p\right)^{1/p} \left(\sum_{n\in A^c} |y(n)|^q\right)^{1/q} \\
&\leq \left(|A|\frac{\epsilon^p}{|A|}\right)^{1/p} \|y\|_q + M\epsilon \\
&= \epsilon\left(\|y\|_q + M\right)
\end{aligned}$$

By making ϵ small enough, we see that $|y(x_k)| \to 0$ as $k \to \infty$.

To see why we require (x_k) to be bounded, consider $p = q = 2$. Take

$$x_k = (0, 0, \ldots, 0, 2^k, 0, \ldots) = 2^k e_k \qquad y = \left(\frac{1}{2}, \frac{1}{2^2}, \ldots, \frac{1}{2^n}, \ldots\right)$$

where x_k is all zeros except in the kth spot. Then we can see that $\lim_k x_k(n) = 0$ for all n, but that for all k,

$$y(x_k) = \sum_n x_k(n)y(n) = 2^k \frac{1}{2^k} = 1$$

□

Problem 6. *Let $f \in C_0(\mathbb{R})$ and for every $t \in \mathbb{R}$ define $f_t \in C_0(\mathbb{R})$ by $f_t(x) = f(x + t)$ for all $x \in \mathbb{R}$.*

(a) Prove that $\{f_t \mid t \in [0,1]\}$ is compact in the norm topology.

Proof. Similar to August 2013 #1

Since $C_c^\infty(\mathbb{R})$ is dense in $C_0(\mathbb{R})$, we can choose $g \in C_c^\infty(\mathbb{R})$ such that $\|g - f\|_\infty < \epsilon$. Then

$$\|f_{t_n} - f_t\|_\infty \le \|f_{t_n} - g_{t_n}\|_\infty + \|g_{t_n} - g_t\|_\infty + \|g_t - f_t\|_\infty$$

It's easy to see $\|f_{t_n} - g_{t_n}\|_\infty$ and $\|g_t - f_t\|_\infty$ are small since $\|g - f\|_\infty < \epsilon$. For $\|g_{t_n} - g_t\|_\infty$, then

$$\|g_{t_n} - g_t\|_\infty = \sup_{x\in\mathbb{R}} |g_{t_n}(x) - g_t(x)| = \sup_{x\in\mathbb{R}} |g(x + t_n) - g(x + t)|$$

where for each fixed $t_n \to t$, since g is compactly supported and continuous then can be sufficiently small for large enough n.

Therefore, the map $G : \mathbb{R} \to C_0(\mathbb{R})$ given by $G(t) = f_t$ is continuous. Since $\{f_t \mid t \in [0, 1]\} = G([0, 1])$ and continuous maps preserve compactness, then the set is compact in the norm topology. □

(b) *Prove that $\{f_t \mid t \in \mathbb{R}\}$ is relatively compact in the weak topology.*

Proof. It suffices to show that every sequence in $A := \{f_t | t \in [0, 1]\}$ has a convergent subsequence in $C_0(\mathbb{R})$. (Nets aren't necessary due to the Eberlein-Šmulian theorem.) Let (f_{t_n}) be a sequence in A.

We may assume $|t_n| \to \infty$. Otherwise (t_n) is frequently in some compact set $[-m, m]$ for $m \in \mathbb{R}_+$. So this subsequence lives in a norm-compact (and hence weak-compact) set by a similar argument from (a).

Recall from Riesz Representation Theorem for $C_0(X)$ that $C_0(\mathbb{R})^* \cong M(\mathbb{R})$. In particular, since $\mathbb{R}$ is σ-compact, for any $\mu \in M(\mathbb{R})$ we have

$$|\mu(\mathbb{R}) - \mu([-m, m])| \to 0.$$

Fix $\varepsilon > 0$ and pick m large enough so that $|\mu(\mathbb{R}) - \mu([-m, m])| < \varepsilon$ *and* that

$$\{x : |f(x)| \ge \varepsilon\} \subset [-m, m]$$

(since $f \in C_0(\mathbb{R})$). Now pick N_m such that $n > N_m$ implies $|t_n| > 2m$. Then

$$\{x : |f_{t_n}(x)| \ge \varepsilon\} \cap [-m, m] = \varnothing.$$

So we then have

$$\begin{aligned} |\int |f_{t_n}(x)|\, d\mu| &\le |\int_{[-m,m]} |f_{t_n}(x)|\, d\mu| + |\int_{[-m,m]^c} |f_{t_n}(x)|\, d\mu| \\ &\le \varepsilon\mu(\mathbb{R}) + \varepsilon M, \end{aligned}$$

where $M := \sup_{x\in\mathbb{R}} |f(x)|$. Letting $\varepsilon \to 0$, we see that f_{t_n} in fact converges weakly to 0. □

Problem 7. *Let f be an arbitrary real valued function on $[0, 1]$. Show that the set of points at which f is continuous is a Lebesgue measurable set.*

Proof. Similar to August 2016, #3.

In fact, we will prove that the set of points at which f is discontinuous is a countable union of closed subsets.

f is continuous at p if for all n, there exists an open U containing p such that $|f(x) - f(y)| < 1/n$ for all $x, y \in U$. Fix n and let

$$V_n = \bigcup_p \{p \text{ s.t. there exists an appropriate } U\} = \bigcup \{\text{appropriate } U\}$$

Hence, V_n is open. Then

$$\{\text{points where } f \text{ is continuous}\} = \bigcap_n V_n$$

So $\{\text{points where } f \text{ is discontinuous}\} = \bigcup_n V_n^c$ where V_n^c is closed. □

Problem 8. *Show that not every nonempty bounded closed subset of ℓ^2 has a point of minimal norm, but that every nonempty bounded closed convex subset of ℓ^2 has a point of minimal norm.*

Proof. Let C be the bounded, closed, convex subset of ℓ^2. Consider the set $\{y \in \mathbb{R} \mid y = \|x\|, x \in C\}$ and since this set is bounded below, there exists an infimum of the set, say s. Then we can find a sequence $x_n \in C$ such that $s \leq \|x_n\| \leq s + \frac{1}{n}$.

I claim that (x_n) is a Cauchy sequence. Indeed, for any $\epsilon > 0$, choose r to be the positive root of the equation $r^2 + 2rs - \frac{\epsilon^2}{4} = 0$.

Since $\|x_n\| \to s$ then there is an N such that $s \leq \|x_n\| < s + r$ for all $n \geq N$. If $n, m \geq N$, then

$$\left\|\frac{x_m - x_n}{2}\right\|^2 = 2\left\|\frac{x_m}{2}\right\|^2 + 2\left\|\frac{x_n}{2}\right\|^2 - \left\|\frac{x_m + x_n}{2}\right\|^2 < \frac{(s+r)^2}{2} + \frac{(s+r)^2}{2} - s^2 = 2sr + r^2 = \frac{\epsilon^2}{4}.$$

So (x_n) is a Cauchy sequence, which means it converges to some x. Since C is closed, $x \in C$ and obtains minimal norm.

Note: This choice of x is unique! If there were two points of minimal norm, say x_1 and x_2 then $\frac{1}{2}(x_1 + x_2) \in C$ by the convexity of C. So $s \leq \left\|\frac{1}{2}(x_1 + x_2)\right\| \leq \frac{1}{2}\|x_1\| + \frac{1}{2}\|x_2\| = s$. Hence, $\|x_1 + x_2\| = 2s$. By the parallelogram law,

$$\|x_1 + x_2\|^2 + \|x_1 - x_2\|^2 = 2\|x_1\|^2 + 2\|x_2\|^2$$

And so $\|x_1 - x_2\|^2 = 4s^2 - 4s^2 = 0$ so $x_1 = x_2$, proving uniqueness.

Counterexample: Consider $M = \left\{\frac{n+1}{n} e_n \mid n \in \mathbb{N}\right\}$. M is closed since the distance between any two of its elements is greater than $\sqrt{2}$ (and thus the only convergent sequences from M are those that are eventually constant). M is clearly non-empty and has no element of minimal norm. □

Problem 9. *Show that there is a sequence $\{f_n\}$ of continuous functions on $[0,1]$ such that* *(a)* $|f_n(t)| = 1$ *for all* n *and all* $t \in [0,1]$ *and*

(b) *for all* $g \in L^1[0,1]$ *one has* $\int_0^1 f_n(t)g(t)dt \to 0$ *as* $n \to \infty$

Proof. It helps to know a bit of Fourier analysis for this one. Let $f_n(t) = e^{int}$. Then we need to show that, for $g \in L^1[0,1]$, $\int_0^1 f_n(t)g(t) \to 0$.

This is surprisingly easy to see for indicator functions. If $g = \mathbb{1}_{[a,b]}$,

$$\int_a^b f_n(t) = \frac{e^{inb} - e^{ina}}{in} \to 0.$$

This extends simply to simple functions, which approximate L^1 functions in L^1-norm. □

Problem 10. *(a) Define what it means for a real valued function on* $[0,1]$ *to be absolutely continuous.*

Proof. The function $f : [0,1] \to \mathbb{R}$ is absolutely continuous if for every $\epsilon > 0$ there exists $\delta > 0$ such that whenever a finite sequence of pairwise disjoint sub-intervals (x_k, y_k) of $[0,1]$ with $x_k, y_k \in [0,1]$ satisfy $\sum_k (y_k - x_k) < \delta$ then $\sum_k |f(y_k) - f(x_k)| < \epsilon$.

Equivalently, f has a derivative f' almost everywhere and the derviative is Lebesgue integrable and for all $x \in [0,1]$,

$$f(x) = f(0) + \int_0^x f'(t)dt.$$

□

(b) Prove that if f *and* g *are absolutely continuous strictly positive functions on* $[0,1]$ *then* f/g *is absolutely continuous on* $[0,1]$.

Proof. Step 1: If f is absolutely continuous, then so is $1/f$.

Since $f > 0$ is continuous on a compact space, there exists some $M \in \mathbb{N}$ such that $\frac{1}{M} \leq |f(x)| \leq M$ for all $x \in [0,1]$.

Indeed, since g is absolutely continuous then for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever a finite sequence of pairwise disjoint sub-intervals (x_k, y_k) of $[0,1]$ with $x_k, y_k \in [0,1]$ satisfy $\sum_k (y_k - x_k) < \delta$ then $\sum_k |f(y_k) - f(x_k)| < \frac{\epsilon}{M^2}$. Then for such intervals, we have

$$\begin{aligned}
\sum \left| \frac{1}{f(y_k)} - \frac{1}{f(x_k} \right| &= \sum \left| \frac{f(x_k) - f(y_k)}{f(y_k)f(x_k)} \right| \\
&\leq \sum \left| \frac{1}{f(y_k)} \right| \left| \frac{1}{f(x_k)} \right| |f(y_k) - f(x_k)| \\
&\leq M^2 \sum |f(y_k) - f(x_k)| \\
&= M^2 \frac{\epsilon}{M^2} = \epsilon.
\end{aligned}$$

Step 2: If f and g are both absolutely continuous, then so is fg.

Find $M \in \mathbb{N}$ such that $|f(x)|, |g(x)| \leq M$ for all $x \in [0,1]$.

Take δ_1 such that if $\sum y_k - x_k < \delta_1$ then $\sum |f(y_k) - f(x_k)| < \epsilon/2M$. Similarly, take δ_2 such that if $\sum y_k - x_k < \delta_2$ then $\sum |g(y_k) - g(x_k)| < \epsilon/2M$. Let $\delta = \min(\delta_1, \delta_2)$. Now

$$\begin{aligned}
\sum |(fg)(y_k) - (fg)(x_k)| &= \sum |f(y_k)g(y_k) - f(x_n)g(x_n)| \\
&\leq \sum |f(y_k)g(y_k) - f(y_k)g(x_k)| + |f(y_k)g(x_k) - f(x_k)g(x_k)| \\
&\leq \sum |f(y_k)||g(y_k) - g(x_k)| + \sum |g(x_n)||f(y_k) - f(x_k)| \\
&\leq M \sum |g(y_k) - g(x_k)| + M \sum |f(y_k) - g(x_k)| \\
&\leq M \frac{\epsilon}{2M} + M \frac{\epsilon}{2M} \\
&= \epsilon
\end{aligned}$$

Combining the two steps, we see immediately that f/g is absolutely continuous. □

27 January 2011

Worth noting before we begin that this is a Johnson qual in January, which means it is particularly challenging in relation to other quals.

Problem 1. *Working directly from thedefinition of almost everywhere convergence, prove that if* $(f_n)_{n=0}^{\infty}$ *is a sequence of measurable functions on a measure space* $(X, \mathcal{M}, \mu)$ *such that*

$$\int_X |f_n - f_0|^{1/4} \, d\mu < n^{-2}$$

for each n*, then* $(f_n)_{n=1}^{\infty}$ *converges to* f_0 μ*-almost everywhere.*

Proof. Let (f_{n_k}) be an arbitrary subsequence of (f_n). Then $|f_{n_k} - f_0|^{1/4} \to 0$ in L^1, so there is a further subsequence $|f_{n_{k_j}} - f_0|^{1/4} \to 0$ a.e. By the commonly used lemma that ($x_n \to x \iff$ for every subsequence x_{n_k} there is a further subsequence $x_{n_{k_j}}$ converging to x), $|f_n - f_0|^{1/4} \to 0$ a.e. Hence so does $|f_n - f_0|$, so $f_n \to f_0$ a.e. □

Problem 2. *Let* K *be a compact metric space. Show that* $C(K)$ *is separable.*

Proof. We separate this into three parts:

(1) Any compact metric space is separable. We have $\bigcup_{x \in X} B(\frac{1}{n}, x) = K$ for all n, so for each n there exists a finite subcover

$$(B(\frac{1}{n}, x_i^n))_{i=1}^{k_n}.$$

We claim the collection $\{x_i^n, i \in [k_n], n \in \mathbb{N}\}$ is dense in K. Let $y \in K$; then for each n $y \in B(\frac{1}{n}, x_{i_y}^n)$ for some $i_y \in [k_n]$. Then $x_{i_y}^n \to y$.

(2) There is a countable collection of continuous functions in $C(K)$ that separates points. Any metric space is Hausdorff, and compact Hausdorff spaces are normal. Define $f_i^n \in C(K,[0,1])$ to be such that

$$f_i^n(\overline{B(\frac{1}{2n},x_i^n)}) \subset \{1\} \quad \text{and} \quad f_i^n|_{B(\frac{1}{n},x_i^n)^c} \equiv 0.$$

Then the collection (f_i^n) separates points. Indeed for any $x,y \in K$ such that $d(x,y) = \varepsilon > 0$, let n be such that $\frac{1}{n} < \varepsilon \leq \frac{1}{n-1}$. Then there exists x_N^i for $N > 2n$ such that $d(x_N^i, x) < \frac{1}{4n}$. So $f_N^i(x) = 1$. But

$$d(x_N^i, y) \geq d(x,y) - d(x_N^i, x) > \frac{1}{n} - \frac{1}{4n} = \frac{3}{4n}.$$

So $f_N^i(y) = 0$.

(3) The rest. The algebra generated by the rational span of 1 and (f_n^i) is countable (clearly the span is countable; then the algebra is generated by taking the countable union of sets $\{f_1 \times \cdots \times f_k : f_j \in (f_n^i)\}$ of cardinality $|\mathbb{N}^{\times k}|$ for $k \in \mathbb{N}$, each of which is countable). What's more, Stone-Weierstrass implies this algebra is dense in $C(K)$. So $C(K)$ is separable. □

Problem 3. *Let $(f_n)_{n=1}^\infty$ be a sequence of measurable functions on a finite measure space $(X, \mathcal{M}, \mu)$. Recall that $(f_n)_{n=1}^\infty$ is said to be uniformly integrable if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$|\int_E f_n \, d\mu| < \varepsilon$$

for all measurable sets $E \subset X$ satisfying $\mu(E) < \delta$ and all n. Prove that if $(f_n)_{n=1}^\infty$ is uniformly integrable, $\sup_n \|f_n\|_1 < \infty$, and $(f_n)_{n=1}^\infty$ converges in measure to 0, then $\|f_n\|_1 \to 0$ as $n \to \infty$.

Proof. Some may know this as a special case of the Vitali Convergence Theorem (see Exercise 15 in Chapter 6 of Folland). Let (f_{n_k}) be a subsequence of (f_n); then there is a further subsequence $(f_{n_{k_j}})$ convergent to 0 a.e. By Egoroff, there is a set A with $\mu(A) < \delta)$ such that $(f_{n_{k_j}})$ uniformly convergenes to 0 uniformly on A^c. Let N be such that

$$\sup_{A^c} |f_{n_{k_j}}| \leq \varepsilon$$

for $k \geq N$. Then

$$\int_X |f_{n_{k_j}}| \, d\mu \leq \varepsilon\mu(X) + \int_A |f_{n_{k_j}}| \overset{\text{Uniform convergence}}{\leq} \varepsilon\mu(X) + \varepsilon.$$

Since our subsequence (f_{n_k}) was arbitrary, $f_n \to 0$ in L^1. □

Problem 4. *Let $1 \leq p < \infty$ and let f be a positive element of $L^p[0,1]$. Prove that the set $\{f^{1/n} : n \in \mathbb{N}\}$ has compact closure in $L^p[0,1]$. Give an example to show that this is false when $p = \infty$.*

Proof. Let us begin with the case where $p = \infty$. Define $f(x) = x$. Then $\|1 - x^{\frac{1}{n}}\|_\infty = 1$ for all n. Hence the set $\{f^{1/n}\}$ is in fact closed. However, the sequence $(f^{1/n})_{n=1}^\infty$ does not have a convergent subsequence.

We now return to where $1 \le p < \infty$. Let $f \in L^p[0,1]$ and define $g := \mathbb{1}_{\{f(x)>0\}}$. We have

$$\int |f^{1/n} - g|^p = \int_{\{0<f(x)<1\}} |f^{1/n} - g| + \int_{\{f(x)\ge 1\}} |f^{1/n} - g|.$$

Note that DCT (or MCT) allows us to pass limits through each integral. So

$$\lim_{n\to\infty} |f^{1/n} - g|^p = \int \lim_{n\to\infty} |f^{1/n} - g|^p.$$

A logarithm argument shows that $\lim_{n\to\infty} f^{1/n} = g$. So g is in the closure of $\{f^{1/n}\}$, and any sequence in $\{f^{1/n}\} \cup \{g\}$ has a convergent subsequence (if no element is in the sequence infinitely many times, then we can find a subsequence of $f^{1/n}$ where n goes to infinity, which converges to g). □

Problem 5. *Let X be a reflexive Banach space and K a non-empty closed convex subset of X. Prove that there exists an $x \in K$ such that $\|x\| = \inf_{y\in K}\|y\|$. Show that this x is unique in the case that X is a Hilbert space.*

Proof. Compare this to Exercise 59 in Chapter 5 of Folland, although we may need to make a slightly different approach for the first part of the problem.

Let $a := \inf_{y\in K}\|y\|$, and take $(y_n) \subset K$ such that $\|y_n\| \to a$. By applying Alaoglu to X^* and noting $X = X^{**}$, there exists a subsequence (y_{n_j}) such that (y_{n_j}) weakly converges to some $y \in X$. (I.e., $(\|y_n\|)$ is bounded by M, and since the M-ball in X^{**} is wk*-compact by Alaoglu, there exists a subsequence of (y_{n_j}) that weak*-converges in the M-ball, which is the same as weakly converging in X.) Now $f(y) = \lim f(y_{n_j})$ for all $f \in X^*$ with $\|f\| = 1$, so

$$\begin{aligned}\|y\| &= \sup_{\|f\|=1} \|f(y)\| = \sup_{\|f\|=1} \lim\|f(y_{n_j})\| \\ &= \lim\|y_{n_j}\| = a.\end{aligned}$$

(It is important that we can give an equality where the indent is; this is possible since for each n_j one can give a norm-one linear functional such that $f(y_{n_j}) = \|y_{n_j}\|$.) Since the y_{n_j} converge in norm to y and K is closed, $y \in K$.

We now assume X is a Hilbert space. We need to use the parallelogram law to guarantee uniqueness. Say that $z, z' \in K$ such that $\|z\| = \inf_{y\in K}\|y\| = \|z'\|$. Then by convexity $\dfrac{z+z'}{2}$ and $\dfrac{z-z'}{2}$ are in K as well. Note the addition of these two elements is z and the subtraction of these two is z'. So

$$\begin{aligned}2\|z\|^2 &= \|z\|^2 + \|z'\|^2 = 2(\|\frac{z+z'}{2}\|)^2 + \|\frac{z-z'}{2}\|^2) \\ \Rightarrow \|z\|^2 &= \|\frac{z+z'}{2}\|^2 + \|\frac{z-z'}{2}\|^2 \ge \|\frac{z+z'}{2}\|^2 \ge \|z\|^2.\end{aligned}$$

This last equality comes from the fact that z has minimum norm in K. So equality holds in this last line. But this means $\|\frac{z-z'}{2}\| = 0 \Rightarrow 0 = z - z' \Rightarrow z = z'$. □

Problem 6. *Let X be a Banach space such that X^* is separable. Prove that X is separable.*

Proof. Let $(f_n)_{n=1}^{\infty}$ be a dense subset of the unit sphere S in X^*. For each $n \in \mathbb{N}$, pick $x_n \in X$ such that $|f_n(x_n)| > \frac{1}{2}$. Let $Y = \overline{\text{span}\{x_n\}}$ and observe that Y is separable (consider finite rational combination of (x_n)). We want to show that $X = Y$; suppose not. Then there is an $f \in S$ such that $f|_Y = 0$. Now the f_n are dense in S, so pick n such that $\|f_n - f\| < \frac{1}{4}$. But $|(f_n - f)(x_n)| > 1/2$ since $f(x_n) = 0$, contradicting our choice of n. □

Problem 7. *(a) State what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be absolutely continuous.*

(b) Let $F : \mathbb{R} \to \mathbb{R}$ be a function and let $0 \leq M < \infty$. Show that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in \mathbb{R}$ iff f is absolutely continuous and $|f'(x)| \leq M$ almost everywhere with respect to Lebesgue measure.

Proof. (a) The definition of absolute continuity can be found at (3.31) in Folland. f is absolutely continuous if $\forall \varepsilon > 0\ \exists \delta > 0$ such that for any finite set of disjoint interval $((a_i, b_i))_{i=1}^N$,

$$\sum_1^N (b_j - a_j) < \delta \Rightarrow \sum_1^N |f(b_j) - f(a_j)| < \varepsilon.$$

(b) ($\Leftarrow$) Say WLOG $y > x$. Then $|f(y) - f(x)| = |\int_x^y f'(t)\,dt| \leq M|y - x|$.

($\Rightarrow$) For any $\varepsilon > 0$, pick $\delta = \frac{\varepsilon}{M}$. Then if $((a_i, b_i))_1^N$ are disjoint intervals such that $\sum_1^N (b_j - a_j) < \delta$, we have

$$\sum_1^N |f(b_j) - f(a_j)| \leq M \sum_1^N |b_j - a_j| < \varepsilon.$$

By the fundamental theorem of calculus, f is differentiable a.e. What's more,

$$|f'(x)| = \lim_{y \to x} \frac{|f(x) - f(y)|}{|x - y|} \leq M.$$

□

Problem 8. *For a function $f : [0, 1] \to \mathbb{R}$ define*

$$\|f\|_L = |f(0)| + \sup\{\frac{|f(x) - f(y)|}{|x - y|} : 0 \leq x < y \leq 1.$$

Prove that the set of all functions $f : [0, 1] \to \mathbb{R}$ satisfying $\|f\|_L < \infty$ is dense in $L^1[0, 1]$.

Proof. Note that these were defined as 1-Hölder continuous functions in Exercise 11 of Chapter 5 in Folland. 1-Hölder continuous functions are indeed (uniformly) continuous; this supremum is the required constant C to see that $|f(x) - f(y)| \leq C|x - y|$. So we would like to show that these are dense in $C[0, 1]$, then apply Theorem 2.26 to say $C[0, 1]$ is dense in $L^1[0, 1]$. (Since $\|\cdot\|_{L^1} \leq \|\cdot\|_{L^\infty}$ on finite measure sets, the L^1-closure of a set in $L^1[0, 1] \cap C[0, 1]$ is contained in the uniform closure; a similar argument is done in working through Exercise 62 of Chapter 5 in Folland.)

To see this, note that continuously differentiable functions are 1-Hölder continuous, as this supremum can be taken to be the max of $|f'|$ on $[0, 1]$. Of course polynomials are continuously differentiable, so we are done. □

Problem 9. *Let $g : [0, 1] \to [0, 1]$ be a continuous function. Determine, with proof, conditions on g which are equivalent to the property that $\lim_{n\to\infty}\|g^n f\|_2 = 0$ for all $f \in L^2[0, 1]$.*

Proof. We know it must be true that $g^n \to 0$ in L^2. This is since

$$\|g^n f\|_2 = \int |f|^2 g^n = \langle |f|^2, g^n\rangle.$$

So if $\lim_n\|g^n f\| = 0$, then $\lim_n g^n$ must be orthogonal to all of L^2. But $C[0, 1] \subset L^2[0, 1]$, so $\lim_n g^n$ must be zero a.e. So we must have g be such that $m(\{g(x) = 1\}) = 0$.

It turns out this is the equivalent condition. WLOG $g(x) < 1$ for all x, and DCT implies $|g^n f|^2 \leq |f|^2$. So

$$\lim_n \int g^n|f|^2 = \int 0|f|^2 = 0.$$

□

Problem 10. *(a) State Fubini's theorem.*

(b) Let $(f_n)_{n=1}^{\infty}$ be a bounded sequence in $C([0, 1]^2)$. Suppose that $f_{x,n} \to 0$ weakly in $L^2(\mu)$ for every $x \in [0, 1]$, where $f_{x,n}(y) = f_n(x, y)$ for all $y \in [0, 1]$ and μ is Lebesgue measure on $[0, 1]$. Prove that $f_n \to 0$ weakly in $L^2(\mu \times \mu)$.

Proof. (a) Fubini's theorem is Theorem 2.37b in Folland. If $f \in L^1(\mu, \nu)$ for σ-finite measures μ, ν, then $f_x \in L^1(\nu)$ for a.e. $x \in X$, $f^y \in L^1(\mu)$ for a.e. $y \in Y$, the a.e.-defined functions $g(x) = \int f_x \, d\nu$ and $h(x) = \int f^y \, d\nu$ are in $L^1(\mu)$ and $L^1(\nu)$, respectively, and

$$\begin{aligned}\int f \, d(\mu \times \nu) &= \int \left[\int f(x, y) \, d\nu(y)\right] d\mu(x) \\ &= \int \left[\int f(x, y) \, d\mu(x)\right] d\nu(y).\end{aligned}$$

(b) Let $M := \sup_n\|f_n\|_\infty$. We want to show that, for any $g \in L^2(\mu \times \mu)$,

$$\int f_n g \, d(\mu \times \mu) \to 0$$

as $n \to \infty$. We have

$$\int |f_n g| \, d(\mu \times \mu) \leq M\|g\|_2,$$

so $f_n g \in L^1$. Fubini applies and we get

$$\begin{aligned}\int f_n g \, d(\mu \times \mu) &= \int \left[\int (f_n g)_x(y) \, d\mu(y)\right] d\mu(x) \\ &= \int \left[\int f_{x,n}(y) g_x(y) \, d\mu(y)\right] d\mu(x).\end{aligned}$$

Since $f_{x,n} \to 0$ weakly in $L^2(\mu)$, this inner integral, which we will call $h_n(x)$, goes to 0 (as $n \to \infty$)

for a.e. x. Fubini also tells us $h_n(x) \in L^1$. Finally, Fubini again tells us that $g_x \in L^2 \subset L^1$, so

$$|\int f_{x,n}(y) g_x(y)| \leq M|\int g_x(y)| \leq M\|g_x\|_1,$$

which is an L^1 constant function in $[0,1]$. Hence DCT applies and

$$\begin{aligned}\lim_{n\to\infty}\int f_n g\, d(\mu\times\mu) &= \lim_{n\to\infty}\int h_n(x)\, d\mu(x) \\ &= \int \lim_{n\to\infty} h_n(x)\, d\mu(x) = \int 0\, d\mu(x) = 0.\end{aligned}$$

Since g was arbitrary, $f_n \to 0$ weakly in $L^2(\mu\times\mu)$. □

28 August 2010

Problem 1. *(a) Give an example of a sequence (f_n) in $L_1[0,1]$ such that $\lim_{n\to\infty}\|f_n\|_{L_1} = 0$, but (f_n) does not converge to 0 almost everywhere.*

(b) Show that if a sequence (f_k) in $L_1[0,1]$ satisfies $\|f_k\|_{L_1} \leq 2^{-k}$ for $k \geq 1$, then $f_k \to 0$ a.e.

Proof. (a) For $n \geq 0$, $1 \leq i \leq 2^n$,

$$f_{2^n+i} = \mathbb{1}_{[\frac{i-1}{2^n},\frac{i}{2^n}]}.$$

Note $\bigcup_{i=1}^{2^n} = [0,1]$ for all n, so for a.e. x $f_k(x) = 1$ for infinitely many k. Yet $\|f_{2n+i}\|_{L_1} = \frac{1}{2n} \to 0$.

(b) We have $\sum_{n=1}^{\infty}\int|f_n| = 1$, so by DCT (Theorem 2.25) $\sum_n f_n$ converges a.e. Hence $f_k = (\sum_{n=1}^{k} f_n - \sum_{n=1}^{k-1} f_n) \to 0$ a.e. □

Problem 2. *Let E be a subset of $[0,1]$ with positve outer Lebesgue measure, i.e. $m^*(E) > 0$. Show that for each $\alpha \in (0,1)$ there is an interval $I \subset [0,1]$ so that*

$$m^*(E\cap I) \geq \alpha m(I).$$

Proof. Compare this to Exercise 30 in Chapter 1 of Folland. The only difference is that E is not necessarily Lebesgue measurable; this may affect some solutions of the exercise.

Suppose not, and let $\alpha \in (0,1)$ such that for any open interval I, $m^*(E\cap I) \leq \alpha m(I)$. By outer measurability, let $U^\varepsilon := \bigcup_{j=1}^{\infty} I_j^\varepsilon$ be an open set such that $m(U^\varepsilon) < m^*(E) + \varepsilon$ for some ε we will choose later. (WLOG one may ask the I_j^ε to be disjoint.) Then

$$\begin{aligned}m^*(E) = m^*(E\cap(\bigcup_{j=1}^{\infty} I_j^\varepsilon) &\leq \sum_{j=1}^{\infty} m(E\cap I_j^\varepsilon) \\ &\leq \alpha m^*(I_j^\varepsilon) = \alpha m^*(U^\varepsilon).\end{aligned}$$

As $\varepsilon \to 0$, $m^*(U^\varepsilon)$ approaches $m^*(E)$. This is a contradiction with the fact that $m^*(E) \leq \alpha m^*(U^\varepsilon)$. □

Problem 3. *Let X be a Banach space and let (x_n) be sequence in x that converges weakly to 0. Prove that $(\|x_n\|)$ is bounded.*

Proof. This is a Uniform Boundedness problem. We have $\|f(x_n)\| \to 0$ for all $f \in X^*$; hence if $\cdot \mapsto \hat{\cdot}$ is the natural inclusion of X into X^{**}, we have

$$\sup_n \|\hat{x}_n(f)\| < \infty.$$

This implies

$$\sup_n \|\hat{x}_n\| < \infty$$

by Uniform Boundedness, and since $\cdot \mapsto \hat{\cdot}$ is an isometry, $(\|x_n\|)$ is bounded. □

Problem 4. *(a) Let (f_n) be a bounded sequence in $C[0,1]$. Prove that*

$$(f_n) \text{ converges weakly to } 0 \iff (f_n) \text{ converges pointwise to } 0.$$

(b) Assume that $(f_n) \subset C[0,1]$ converges in the weak topology. Show that f_n is norm convergent in $L_1[0,1]$. [For part (b) you may use problem (3).]

Proof. (a) ($\Rightarrow$) For $x \in [0,1]$, define $\hat{x}(f) = f(x)$. Since $\hat{x}(\lambda f + g) = \lambda f(x) + g(x) = \lambda \hat{x}(f) + \hat{x}(g)$, $\hat{x}$ is linear. So $\widehat{[0,1]} \subset C[0,1]^*$. Hence if (f_n) converges weakly to 0, so does $\hat{x}(f) = f(x)$ for $x \in [0,1]$.

($\Leftarrow$) We use the Riesz representation for $C[0,1]$. Suppose (f_n) converges pointwise to 0 and let μ be a complex Radon measure for $[0,1]$. Of course $\mu([0,1]) < \infty$ since $[0,1]$ is compact. In particular, if $|f_n| \leq M$ for all n for some $M > 0$, $M1 \in L^1(\mu)$, so DCT applies and

$$\lim_{n\to\infty} \int f_n \, d\mu = \int 0 \, d\mu.$$

So if $\phi_\mu \in C[0,1]^*$ is the corresponding state, $\phi_\mu(f_n) = \int f_n \, d\mu \to 0$.

(b) Let f be the weak limit of f_n (in particular, $f \in C[0,1]$); then $(f_n - f)$ converges weakly to 0 in $C[0,1]$. Hence by (3) $(\|f_n - f\|_\infty)$ is bounded, say by M. So (4a) applies and $(f_n - f)$ converges pointwise to 0. Now Egoroff gives us a set E_ε such that $m(E_\varepsilon) < \varepsilon$ and $f_n - f$ converges uniformly on E_ε^c. Then

$$\lim_{n\to\infty} \int |f_n - f| \, dm \leq \lim_{n\to\infty} \int_{E_\varepsilon^c} |f_n - f| \, dm + M\frac{\varepsilon}{2}$$

In particular, if n is large enough so that $|f_n - f| < M\frac{\varepsilon}{2}$ on E_ε^c, we get

$$\lim_{n\to\infty} \int |f_n - f| \, dm \leq M\varepsilon.$$

Letting $\varepsilon \to 0$, we are done. □

Problem 5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function such that for some $C > 0$*

$$m\{x : |f(x)| \geq \lambda\} \leq C\lambda^{-2}, \text{ for all } \lambda > 0.$$

Prove that there is some $C' > 0$ so that

$$\int_e |f(x)|\, dx \leq C'\sqrt{m(E)}, \text{ for all measurable } E \subset \mathbb{R}.$$

Proof. If $m(E)$ is infinite, this trivially holds, so we assume $m(E) < \infty$. We note the fact that

$$m(\{x \in E : |f(x)| \geq y\}) \leq \min(\frac{C}{y^2}, m(E)).$$

Let $N = \dfrac{C^{1/2}}{(m(E))^{1/2}}$. Then

$$\begin{aligned}
\int_E |f(t)|\, dt &= \int_E \int_{y \leq |f(t)|} dy\, dt \\
&\overset{\text{Tonelli}}{=} \int_0^\infty \int_{\substack{|f(t)| \geq y \\ t \in E}} dt\, dy \\
&= \int_N^\infty m\{t \in E : |f(t)| \geq y\}\, dy + \int_0^N m\{t \in E : |f(t)| \geq y\}\, dy \\
&\leq \int_N^\infty \frac{C}{y^2}\, dy + \int_0^N m(E)\, dy \\
&= C[-\frac{1}{y}]_N^\infty + m(E)N \\
&= C^{1/2}m(E)^{1/2} + m(E)^{1/2}C^{1/2} =: C'm(E)^{1/2}.
\end{aligned}$$

□

Problem 6. *Let $f(x)$ be a continuous function on $[0,1]$ with a continuous derivative $f'(x)$. Given $\varepsilon > 0$, prove that there is a polynomial $p(x)$ so that*

$$\|f(x) - p(x)\|_\infty + \|f'(x) - p'(x)\|_\infty < \varepsilon.$$

Proof. Use Stone-Weierstrass to find a polynomial q such that $\|f' - q\|_\infty < \frac{\varepsilon}{2}$. Let p be a polynomial such that $p' = q$ and $p(0) = f(0)$. Then since $f(x) = \int_0^x f'(x)\, dx + f(0)$ and $p(x) = \int_0^x q(x)\, dx + p(0)$, we get

$$\begin{aligned}
\sup_{x \in [0,1]} |f(x) - p(x)| &= \sup_{x \in [0,1]} |\int_0^x f' - q\, dm| \\
&\leq \|f' - q\|_\infty m([0,1]) = \frac{\varepsilon}{2}.
\end{aligned}$$

□

Problem 7. *Let X be a non-empty complete metric space and let*

$$(f_n : X \to \mathbb{R})_{n=1}^\infty$$

be a sequence of continuous functions with the following property: for each $x \in X$, there exists an integer N_x so that $(f_n(x))_{n \geq N_x}$ is either a monotone increasing or decreasing sequence. Prove that there is a non-empty open subset $U \subset X$ and an integer N so that the sequence $(f_n(x))_{n \geq N}$ is monotone for all $x \in U$.

Proof. With this contorted language, this can't *not* be a Baire Category Theorem problem. Define

$$X_N := \{x \in X : (f_n(x))_{n \geq N} \text{ is monotonically increasing or decreasing}\}.$$

It is not hard to show that X_n is closed: if $x_n \to x$ such that $f_{m+1}(x_n) \geq f_m(x_n)$ for all n, then since $f_{m+1}(x_n) \to f_m(x)$ and $f_m(x_n) \to f_m(x)$, it must be true that $f_{m+1}(x) \geq f_m(x)$.

We now show that, if there exists an open set V an integer N such that $(f_n(x))_{n \geq N}$ is monotonically increasing or decreasing for all $x \in V$, then there is a set U as in the problem (i.e., using the same integer N, f_n is eventually monotonic in one direction only on U). Take $X_1' := \{x \in X : f_n(x)$ is eventually monotonically increasing$\}$ and $X_2 := (X_1)^c$. The union of these is X, and by a similar argument to the above each are also closed, so by Baire Category Theorem one of these contains an open set, proving our claim.

Hence if such a U as above doesn't exist, such a V doesn't either, so the X_N's are nowhere dense and Baire Category Theorem (again, again!) completes the proof. □

Problem 8. *Assume that $1 \leq p < \infty$ and that a linear operator $T : L_p[0,1] \to L_p[0,1]$ is such that (Tf_n) converges almost everywhere to 0 if (f_n) converges almost everywhere to 0. Show that T is a bounded operator on $L_p[0,1]$.*

Proof. This problem's solution may use (maybe unexpectedly) the Closed Graph Theorem. Suppose $(f_n, Tf_n) \to (0, h)$ in the graph of $L_p[0,1]^2$. We want to show $h = T(0)$, since then CGT would imply T_k is continuous at 0 and hence is bounded.

We have $f_n \to 0$ in L_p. Then $f_n^p \to 0$ in L_1, so there is a subsequence $f_{n_k}^p$ that converges to 0 a.e. Hence $f_{n_k} \to 0$ a.e.

By similar reasoning there is a further subsequence $f_{n_{k_j}}$ such that $Tf_{n_{j_k}} \to h$ a.e. By assumption, $Tf_{n_{j_k}} \to 0$ a.e., so $h = 0$ a.e. Therefore $h = T(0) = 0$. □

Problem 9. *(a) State the Hahn-Banach Theorem for real vector spaces.*

(b) Deduce from it the following corollary: Let X be a Banach space, $Y \subset X$ a closed subspace and $x \in X \backslash Y$. Show that there is an $x^* \in X^*$ such that $x^*|_Y \equiv 0$ and $x^*(x) = 1$.

Proof. See Section 5.2 of Folland, noting especially Theorem 5.8a. □

Problem 10. *Let U be the closed unit ball in the Banach space $C[0,1]$ of continuous real valued functions on the unit interval. Prove that the extreme points of U are the constant functions ± 1. Prove that $C[0,1]$ is not a dual Banach space.*

Proof. Define the functions

$$g(x) := f + \frac{1}{2}(1 - |f|), \quad h(x) := f - \frac{1}{2}(1 - |f|).$$

Then $f = \frac{1}{2}(g+h)$. Note that if $f \neq \pm 1$, $g \neq f \neq h$, so we only need show $\|g\|_\infty, \|h\|_\infty \leq 1$.

$$|g(x)| = \frac{1}{2}|f(x)| + \frac{1}{2} \leq 1,$$
$$|h(x)| = \frac{3}{2}|f(x)| - \frac{1}{2} \leq 1.$$

Now clearly if $1 = \frac{1}{2}f + \frac{1}{2}g$ for $f, g \in C[0,1]$, then $f(x) < 1 \Rightarrow g(x) > 1$, which cannot happen. So $f \equiv 1 \equiv g$, and ± 1 are the extreme points of U.

Now if $C[0,1]$ is a dual Banach space, Alaoglu implies its unit ball is weak*-compact. Since this space is clearly convex, Krein-Milman would imply that $U = \lambda 1$ for $\lambda \in [-1,1]$, which is clearly untrue. □

29 January 2010

Problem 1. *Is it possible to find uncountably many disjoint measurable subsets of* $\mathbb{R}$ *with strictly positive Lebesgue measure?*

Proof. No! We first test to see whether this might be true on $[-n,n]$ for $n \in \mathbb{N}$. Say (E_α) is some collection of disjoint measurable sets contained in $[-n,n]$. Then

$$\sum_\alpha m(E_\alpha) \leq m([-n,n]) < \infty,$$

and an elementary argument shows that $\#\{\alpha : m(E_\alpha) > 0\}$ is countable. (For those who have not seen it before: we have $\{\alpha : m(E_\alpha) > 0\} = \bigcup_{n=1}^\infty \{\alpha : m(E_\alpha) > \frac{1}{n}\}$, and each of these latter sets must be finite since the sum is finite.)

We now apply a very similar argument for (E_α) contained in $\mathbb{R}$: in particular, $m(E_n) = \bigcup_{n=1}^\infty m(E_n \cap [-n,n])$ by continuity from below, so $\{\alpha : m(E_\alpha) > 0\} = \bigcup_{n=1}^\infty \{\alpha : m(E_\alpha \cap [-n,n]) > 0\}$. By the above paragraph each of the sets in this union is countable, so $\{\alpha : m(E_\alpha) > 0\}$ is countable. □

Problem 2. *Let* f *be a non-negative element of* $L_1[0,1]$. *Prove that*

$$\lim_{n\to\infty} \int_0^1 \sqrt[n]{f(x)}\, dx = m(\{x : f(x) > 0\}).$$

Proof. We can split this integral into

$$\int_{\{x:f(x)\leq 1\}} \sqrt[n]{f(x)}\, dx + \int_{\{x:f(x)>1\}} \sqrt[n]{f(x)}\, dx =: A + B.$$

DCT applies to both, with 1 being the dominating function for A and f being the dominating function for B. (MCT also works fine.) So

$$\lim_{n\to\infty} \int_0^1 \sqrt[n]{f(x)}\, dx = \int_0^1 \lim_{n\to\infty} \sqrt[n]{f(x)}\, dx.$$

Note

$$\log \lim_{n\to\infty} f(x)^{1/n} = \lim_{n\to\infty} \frac{1}{n} \log f(x) = 0 \quad \text{when} \quad f(x) \neq 0,$$

so $\lim_{n\to\infty} \sqrt[n]{f(x)} = 1$ when $f(x) \neq 0$. When $f(x) = 0$, $\sqrt[n]{f(x)} = 0$ for all n, so

$$\lim_{n\to\infty} \int_0^1 \sqrt[n]{f(x)}\, dx = \int_0^1 1_{\{x:f(x)>0\}}\, dx = m(\{x : f(x) > 0\}).$$

□

Problem 3. *(a) Let X be a Banach space with a closed subspace E. If $x \in X$, prove that there exists $\phi \in X^*$ such that $\|\phi\| = 1$, $\phi|_E = 0$, and*

$$\phi(x) = \operatorname{dist}(x, E).$$

(b) Taking $X = C[-1, 1]$ and E to be the subspace of even functions ($f(t) = f(-t)$), consider an odd function $g \in X$ ($g(-t) = -g(t)$). Prove that there exists $\phi \in X^*$, $\|\phi\| = 1$, $\phi|_E = 0$, and

$$\phi(g) = \|g\|_\infty.$$

Proof. (a) Compare this problem with Theorem 5.8a. Define $\phi : E + \lambda x \to \mathbb{C}$ to be

$$\phi(e + \lambda x) = \lambda \operatorname{dist}(x, E).$$

Then $\phi|_E = 0$, $\phi(x) = \operatorname{dist}(x, E)$, and

$$\begin{aligned} |\phi(e + \lambda x)| &\leq |\lambda| \inf\{\|x + e'\| : e' \in E\} \\ &\leq |\lambda| \|\lambda^{-1} e + x\| = \|e + \lambda\|, \end{aligned}$$

a sublinear functional. We have also shown that $\|\phi\| \leq 1$; to see $\|\phi\| = 1$, take $\lambda = 1$ and take a sequence of e' that approximates $\operatorname{dist}(x, E)$. Hahn-Banach completes the proof.

(b) Due to (a), we need to only show a couple things. First, we note that E is closed. This comes directly from continuity of the norm: $\|h(x) - h(-x)\| = \lim_{n\to\infty} \|h_n(x) - h_n(-x)\| = 0$.

Second, we want to show: if f is even and g is odd,

$$\|f - g\|_\infty \geq \|g\|_\infty.$$

If $|(f - g)(x)| \leq |g(x)|$, then $|(f - g)(-x)| = |f(x) + g(x)|$. Since $2|g(x)| \leq |(f - g)(x)| + |(f + g)(x)|$, we get $|(f - g)(-x)| = |(f + g)(x)| \geq |g(x)|$. Hence $\sup_{y\in[0,1]} |(f - g)(y)| \geq |g(x)|$ for all $x \in [0, 1]$, so $\|f - g\|_\infty \geq \|g\|_\infty$. The reverse inequality comes from choosing $f \equiv 0$: $\|0 - g\|_\infty = \|g\|_\infty$. Hence $\operatorname{dist}(g, E) = \|g\|_\infty$ and the rest is due to (a). □

Problem 4. *Let m be Lebesgue measure on $[0, 1]$. If $(f_k)_{k=1}^\infty$ and $(g_k)_{k=1}^\infty$ are orthonormal bases for $L_2([0, 1], m)$, prove that $(f_k(x)g_\ell(y))_{k,\ell=1}^\infty$ is an orthonormal basis for $L_2([0, 1] \times [0, 1], m \times m)$.*

Proof. Compare this to Exercise 61 in Chapter 5 of Folland.

First, we want to show that this set is orthonormal. Note

$$\begin{aligned}\langle f_k g_\ell, f_k g_\ell\rangle &= \int |\bar{f}_k f_k|^2\,dx \int |\bar{g}_\ell g_\ell|^2\,dy\\ &= \|f_k\|^2\|g_k\|^2 = 1.\end{aligned}$$

And if $k \neq k'$ [resp. $\ell \neq \ell'$],

$$\langle f_k g_\ell, f_{k'} g_{\ell'}\rangle = \langle f_k, f_{k'}\rangle\langle g_\ell, g_{\ell'}\rangle = 0$$

(we are using the common trick here of using Tonelli to show $f_k g_\ell \overline{f_{k'} g_{\ell'}} \in L_1$, then using Fubini to get the equality here). So this is an orthonormal set. (We have also shown that $f_k g_\ell \in L^2([0,1]\times[0,1])$.)

Now assume $h \in L^2([0,1]\times[0,1])$ is a function such that

$$\langle h(x,y), f_k(x) g_\ell(x)\rangle = 0 \quad \forall k, \ell.$$

Then Fubini applies to the below since these functions are in L^1 due to Hölder:

$$\begin{aligned}&\int_{[0,1]^2} h(x,y)\bar{f}_k(x)\bar{g}_\ell(y) = \int \bar{g}_\ell[\int \bar{f}_k h^y(x)\,dx]\,dy = 0\\ &\overset{g_\ell \text{ onb}}{\Rightarrow} \int \bar{f}_k h^y(x) = 0\\ &\overset{f_k \text{ onb}}{\Rightarrow} h^y(x) = h(x,y) = 0,\end{aligned}$$

where k, ℓ above are arbitrary. So $(f_k g_\ell)$ is complete and hence an orthonormal basis. □

Problem 5. *In $C[0,1]$, let*

$$A = \operatorname{span}\{x^n(1-x) : n \geq 1\}.$$

Prove that A is an algebra whose uniform closure is

$$\{f \in C[0,1] : f(0) = f(1) = 0\}.$$

Proof. A main consideration is determining how to apply Stone-Weierstrass whenever our functions are equivalently zero at *two* points. One way to do it is to consider the one-point compactification of $[0,1)$. Here is another: there exists a natural homeomorphism between $\mathbb{T}$ and $[0,1]/\{0,1\}$ (namely, $x \mapsto e^{2\pi i x}$), which induces an isometric algebra isomorphism $C(\mathbb{T}) = \{f \in C[0,1] : f(0) = f(1)\}$. $\mathbb{T}$ is a compact Hausdorff space as it is a closed subset of $\mathbb{C}$, so it suffices to show that

$$\operatorname{span}\{x^n(1-x) : n \geq 1\}$$

is an algebra that separates points. (It is closed under conjugation by definition of span; note our domain is in $\mathbb{R}$.)

We note

$$x^n(1-x)x^m(1-x) = x^{n+m}(1-x) - x^{n+m+1}(1-x),$$

so the span of these elements are closed under multiplication, which is enough to show this span is an algebra. Note that

$$f_k(x) = \sum_{n=1}^{k} x^n - x^{n+1}$$

approximates the polynomial x uniformly on any closed interval not including 0=1, so if $a \neq b$ for $a, b \notin \{0\}$ there is some k such that f_k separates them. Clearly $f(0) = 0$ for all $f \in A$, and for $a \neq 0$ $f_k(a)$ is eventually nonzero.

Hence Stone-Weierstrass applies and this algebra equals

$$\{f \in C(\mathbb{T}) : f(1) = 0\} = \{f \in C[0,1] : f(0) = f(1) = 0\}.$$

□

Problem 6. *(a) State the Riesz representation theorem for the dual of $L_p(\mu)$, where μ is a σ-finite measure on some measurable space (Ω, Σ, μ), and $1 \leq p < \infty$.*

(b) Prove the following part of the above theorem (you can assume μ is finite): Let $f \in L_p(\mu)^*$. Then there is a $g \in L_1(\mu)$ so that

$$\int_a g\, d\mu = F(\chi_A),$$

for all $A \in \Sigma$.

Proof. Compare this to Theorem 6.15 in Folland.

(a) This theorem states the following for σ-finite μ: "Let p and q be conjugate exponents. If $1 \leq p < \infty$, for each $\phi \in (L^p)^*$ there exists $g \in L^q$ such that $\phi(f) = \int fg$ for all $f \in L^q$. and hence L^q is isometrically isomorphic to $(L^p)^*$.

(b) First let us suppose that μ is finite, so that all simple functions are in L^p. If $\phi \in (L^p)^*$ and E is a measurable set, let $\nu(E) = \phi(\chi_E)$. For any disjoint sequence (E_j), if $E = \bigcup_1^\infty E_j$ we have $\chi_E = \sum_1^\infty \chi_{E_j}$ where the series converges in the L^p norm:

$$\|\chi_E - \sum_1^n \chi_{E_j}\|_p = \|\sum_{n+1}^\infty \chi_{E_j}\|_p = \mu(\bigcup_{n+1}^\infty E_j)^{1/p} \to 0 \text{ as } n \to \infty.$$

Hence, since ϕ is linear and continuous,

$$\nu(E) = \sum_1^\infty \phi(\chi_{E_j}) = \sum_1^\infty \nu(E_j),$$

so that ν is a complex measure. Also, if $\mu(E) = 0$, then $\chi_E = 0$ as an element of L^p, so $\nu(E) = 0$; that is, $\nu \ll \mu$. By the Lebesgue-Radon-Nikodym theorem there exists $g \in L^1(\mu)$ such that $\phi(\chi_E) = \nu(E) = \int_E g\, d\mu$ for all E. □

Problem 7. *Let $1 < p < \infty$ and $f \in L_p[0, \infty)$. Show that*

$$a) \quad |\int_0^x f(t)\, dt| \leq \|f\|_p x^{1-\frac{1}{p}}, \text{ for } x > 0.$$

$$b)\qquad \lim_{n\to\infty}\frac{1}{x^{1-\frac{1}{p}}}\int_0^x f(t)\,dt = 0.$$

Hint for part b): first assume that f as compact support.

Proof. a) We have

$$\begin{aligned}|\int_0^x f(t)\,dt| &\le \int |f\mathbb{1}_{[0,x]}|\,dt = \|f\mathbb{1}_{[0,x]}\| \\ &\overset{\text{Hölder}}{\le} \|f\|_p\|1_{[0,x]}\|_{1-\frac{1}{p}} = \|f\|_p x^{1-\frac{1}{p}}.\end{aligned}$$

b) If f has compact support, there is some $x_0 > 0$ such that $f(x) = 0$ for $x > x_0$. Then

$$\lim_{x\to\infty}|\frac{1}{x^{1-\frac{1}{p}}}\int_0^x f(x)\,dt| \le \lim_{x\to\infty}\|f\|_p\left(\frac{x_0}{x}\right)^{1-\frac{1}{p}} = 0.$$

Now for arbitrary $f \in L_p$ and $\varepsilon > 0$, there is some $x_0 > 0$ such that $\int_{x_0}^\infty |f|^p < \varepsilon$. Then for $x \gg x_0$,

$$\begin{aligned}\lim_{x\to\infty}|\frac{1}{x^{1-\frac{1}{p}}}\int_0^x f(x)\,dt| &\le \lim_{x\to\infty}|\frac{1}{x^{1-\frac{1}{p}}}\int_0^{x_0} f(x)\,dt| + \lim_{x\to\infty}|\frac{1}{x^{1-\frac{1}{p}}}\int_{x_0}^x f(x)\,dt| \\ &\overset{(a)}{\le} \lim_{x\to\infty}\frac{1}{x^{1-\frac{1}{p}}}\|f1_{[x_0,\infty)}\|_p x^{1-\frac{1}{p}} = 0.\end{aligned}$$

□

Problem 8. *Let X be a finite-dimensional vector space.*

(a) If $\|\cdot\|$ is a norm on X, prove that $(X, \|\cdot\|)$ is complete.

(b) If $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on X, prove that there exist constants $c, k > 0$ such that

$$c\|x\|_1 \le \|x\|_2 \le k\|x\|_1, \quad x \in X.$$

(Hint: note that without loss of generality $\|\cdot\|_1$ can be chosen to be, say, the ℓ_1-norm with respect to some basis.)

Proof. Let $(e_i)_1^n$ be a basis for X.

(a) For the first statement it suffices to show that, when $(\sum_{i=1}^n a_i^j e_i)_j \subset X$ such that

$$\sum_{j=1}^\infty \|\sum_{i=1}^n a_i^j e_i\| < \infty,$$

$\sum_{j=1}^{\infty}\sum_{i=1}^{n} a_i^j e_i$ exists in X. We have

$$\infty > \sum_{j=1}^{\infty}\|\sum_{i=1}^{n} a_i^j e_i\| \geq \|\lim_{k\to\infty}\sum_{i=1}^{n} e_i \sum_{j=1}^{k} a_i^j\|$$
$$= \|\sum_{i=1}^{n} e_i(\sum_{j=1}^{\infty} a_i^j)\|.$$

By (b) (which we will prove without using (a)), all norms on X are equivalent, so we may consider $\|\cdot\|$ to be the ℓ_1-norm. Hence $\infty > \sum_{i=1}^{n}|\sum_{j=1}^{\infty} a_i^j| \geq \sum_{j=1}^{\infty} a_i^j|$ for all i. Define $a_i := \sum_{j=1}^{\infty} a_i^j$; then $\sum_{i=1}^{n} a_i e_i$ is the limit of this convergent sum, so X is complete.

(b) Compare to Exercise 6d of Chapter 5 in Folland; we will take the other statements in this exercise for granted, although we will show where we use the other parts. Let $\|\cdot\|_1$ be the ℓ_1-norm on X. Let $\|\cdot\|_2$ is another norm on X and define $C := \{x \in X : \|x\|_1 = 1\}$, which is compact by an exercise in Folland (6c). Then $\|\cdot\|_2 : C \to \Bbbk$ is continuous (by Exercise 6b in Folland) and hence bounded (since C is compact). Let M_1 be such that $\|x\|_2 \leq M_1 = M_1\|x\|_1$ for $x \in C$. Then by linearity of the norm we have $\frac{1}{M_1}\|\cdot\|_2 \leq \|\cdot\|_1$.

Now we claim there exists $M_2 > 0$ such that $\|x\|_2 \geq M_2$ for all $x \in C$. If not there is some sequence $(x_n) \subset C$ such that $\|x_n\|_2 \to 0$. But again, C is compact, so $0 \in C$, contradiction. Hence such an M_2 exists and, by linearity of the norm again, $\|\cdot\|_1 \leq M_2\|\cdot\|_2$. Setting $c = \frac{1}{M_1}$ and $k = M_2$, we are done. □

Problem 9. *Let P be the vector space of all polynomials with real coefficients. Show there is no norm on P which turns P into a Banach space. (Hint: you may use the first statement of Problem 8 even if you have been unable to prove it.)*

Proof. This is a Baire Category Theorem problem, as is common in problems that ask you to prove something cannot be a Banach space - there is usually a structural problem coming from how norms (hence metrics) interact with the space.

Say there is such a norm $\|\cdot\|$ that turns P into a Banach space. Define $P_n := \{a_0 + a_1x + \cdots + a_nx^n : a_i \in \mathbb{R}, i \in [n]_0\}$. We claim these spaces are nowhere dense. Indeed, say $p := a_0 + a_1x + \cdots + a_nx^n$ is an arbitrary element of P_n. Then for $\varepsilon > 0$, the polynomial $p' = p + \frac{\varepsilon}{2\|x^{n+1}\|}x^{n+1} \in B(\varepsilon, p)$, so P_n must have empty interior. Also, P_n is closed: we known P_n is a finite-dimensional vector space spanned by $\{1, x, \ldots, x^n\}$, so by 8a) it is complete and hence closed. So these sets are nowhere dense and Baire Category Theorem gives us our contradiction. □

Problem 10. *Let $p \in [1,\infty)$. Show that the unit ball of $L_\infty[0,1]$ is weakly closed in $L_p[0,1]$.*

Proof. First $L_\infty[0,1] \subset L_p[0,1]$, so the unit ball of $L_\infty[0,1]$ is contained in $L_p[0,1]$. Say $f_n \subset B(L_\infty[0,1])$ such that

$$g(f_n) \to g(f) \Rightarrow \int f_n g \to \int fg$$

for all $g \in L_p^* = L_q$. In particular, if $g = 1_A$ for any Borel set $A \subset [0,1]$, we see that

$$\int_A f_n \to \int_A f.$$

Now we see that $|\frac{1}{m(A)}\int_A f_n| \leq 1$ for all Borel sets A with positive measure, so $|\frac{1}{m(A)}\int_A f| \leq 1$ as well. Lebesgue Differentiation theorem then says $|f| \leq 1$ a.e., so $f \in B(L_\infty[0,1])$.

Alternatively: set $A_n := \{x : \mathrm{Re}(f) \geq 0, \mathrm{Im}(f) \geq 0, |f| > 1 + \frac{1}{n}\}$. Then this inequality above guarantees that $m(A_n) = 0$ for all n, and hence so is $\{x : \mathrm{Re}(f) \geq 0, \mathrm{Im}(f) \geq 0, |f| > 1\} = \bigcup_{n=1}^\infty A_n$. The process is similar for showing that $\{x : \mathrm{Re}(f) \geq 0, \mathrm{Im}(f) < 0, |f| > 1\}$ and the other two corresponding sets when $\mathrm{Re}(f) \leq 0$ are zero-measure sets. From this we see that $\|f\|_\infty \leq 1$ and $f \in B(L_\infty[0,1])$. □

30 August 2009

Problem 1. *Evaluate the iterated integral*

$$\int_0^\infty \int_0^\infty x \exp(-x^2(1+y^2))\, dx\, dy.$$

(Justify your answer.)

Proof. *(Thanks to the Qual Prep course of 2022 for spotting the solution, as the compiler could not.)* For the inner integral, we use u-substitution with $u = -x^2(1+y^2)$:

$$\int_0^\infty x \exp(-x^2(1+y^2))\, dx = \frac{1}{2(1+y^2)},$$

so this iterated integral is equal to

$$\frac{1}{2}\int_0^\infty \frac{1}{1+y^2}\, dy = \frac{1}{2}\frac{\pi}{2} = \frac{\pi}{4}.$$

□

Problem 2. *Let $f \in C[0,1]$ be real-valued. Prove that there is a monotone increasing sequence of polynomials $(p_n(x))_{n=1}^\infty$ converging uniformly on $[0,1]$ to $f(x)$.*

Proof. The first part of this problem can be gotten through Stone-Weierstrass treatment; to show we can construct such a sequence to be monotone increasing will take a bit of work.

We note that $[0,1]$ is compact Hausdorff, the collection of polynomials $\mathbb{P}$ is indeed a unital algebra, and that $x - x_0$ separates the point x_0 away from any other point. Since this algebra is closed under complex conjugation, Stone-Weierstrass tells us the collection of polynomials $\mathbb{P}$ is uniformly dense in $C[0,1]$.

Take $f \in C[0,1]$. For any $n \in \mathbb{N}$, we note that we may find a polynomial $p_n \in \mathbb{P}$ such that

$$\|(f - \frac{1}{n}) - p_n\|_\infty < \frac{1}{2}(\frac{1}{n} - \frac{1}{n+1}).$$

Note that $(f - \frac{1}{n+1}) - (f - \frac{1}{n}) = \frac{1}{n} - \frac{1}{n+1}$, so this condition guarantees that $p_{n+1} \geq p_n$. What's more,

$$\|f - p_n\| \leq \frac{1}{n} + \frac{1}{2n(n+1)},$$

and this latter expression converges to zero as $n \to \infty$. So (p_n) is the desired sequence. □

Problem 3. *Let (f_n) be a sequence of non-zero elements of $L^2[0,1]$. Prove that there is a function $g \in L^2[0,1]$ such that for all $n \geq 1$ we have*

$$\int_0^1 g(x) f_n(x)\,dx \neq 0.$$

Proof. This is a Baire category theorem problem. The things that lead this way: we have a "sequence" (i.e., countable number) of elements with which g must have nonzero L^2-inner product with at least one. This seems to suggest a contradiction proof, where we might be able to build a countable collection of nowhere dense sets. (Recall that $\langle f_n, g\rangle_{L^2} = \int_0^1 \bar{g} f_n\,dx$.)

To the end we suppose there does not exist such a function g. Then for all $g \in L^2[0,1]$ there must exist an n such that $\langle f_n, g\rangle = 0$. Hence $g \in \langle f_n\rangle^\perp$, the orthogonal complement of the (closed, since finite-dimensional) subspace generated by f_n. This is equivalent to the statement

$$L^2[0,1] = \bigcup_{n=1}^\infty \langle f_n\rangle^\perp.$$

These sets $\langle f_n\rangle^\perp$ are nowhere dense. To see this, first note that this set is closed by continuity of the inner product. Fix $\varepsilon > 0$. For any $g \in \langle f_n\rangle^\perp$, note

$$\|(g + \varepsilon f_n) - g\|_2 < \varepsilon \|f_n\|_2,$$

and $\langle f_n, g + \varepsilon f_n\rangle = \varepsilon\langle f_n, f_n\rangle > 0$. So $g + \varepsilon f_n \notin \langle f\rangle^\perp$, yet is in any $\varepsilon\|f_n\|_2$-ball around g.

Baire Category Theorem finishes the proof. □

Problem 4. *Let (X, Σ, μ) be a measure space with $\mu(X) < \infty$. Given set $A_i \in \Sigma, i \geq 1$, prove that*

$$\mu(\bigcap_{i=1}^\infty A_i) = \lim_{n\to\infty} \mu(\bigcap_{i=1}^n A_i).$$

Give an example to show that this need not hold when $\mu(X) = \infty$.

Proof. Setting $B_n := \bigcap_{i=1}^n A_i$, this is guaranteed by applying continuity from above to the collection of (B_n). (See Folland Theorem 1.8c,d for a proof; it is usually preferred to give a proof more than one sentence long, so you might prove continuity from above using continuity from below using the fact that our measure space is finite.)

For the example we consider $A_i := [i, \infty)$ in $\mathbb{R}$. Note $\bigcap_{i=1}^\infty A_i = \varnothing$, so $\mu(\bigcap_{i=1}^\infty A_i) = 0$, but

$$\mu(\bigcap_{i=1}^n A_i) = \mu([n, \infty)) = \infty.$$

□

Problem 5. *Let K be a compact subset of $\mathbb{R}^n$ and describe the dual space of the Banach space $C(K)$. (You may choose either the real or the complex Banach space.)*

Let $\mathbb{1} \in C(K)$ denote the constant function taking value 1 and let S be the subset of the dual space consisting of the positive bounded linear functionals on $C(K)$ that map $\mathbb{1}$ to 1. Show that the extreme points of S are the point evaluation maps, $f \mapsto f(x)$.

Proof. The choice of real or complex Banach space makes not much difference. In the case of the complex Banach space, Riesz representation theorem for $C(X)$ identifies the dual space of $C(K)$ with the space of complex Radon measures on K. In the case of the real Banach space, these are replaced with the space of signed Radon measures on K.

For the second problem, we quickly note that these unital positive bounded linear functionals correspond with probability Radon measures $\text{Prob}(K)$ (i.e., positive measures μ such that $\mu(K) = 1$). Also, the point evaluation maps $f \mapsto f(x)$ correspond to the single atomic measures

$$\delta_x(E) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}.$$

So we want to show the single atomic measures are the extreme points in $\text{Prob}(K)$. These are indeed extreme points, since if $\delta_x = t\mu + (1-t)\nu$ for $t \in (0,1)$ and $\mu, \nu \in \text{Prob}(K)$, we must have $\mu(\{x\}^c) = \nu(\{x\}^c) = 0$ by positivity, so μ and ν must be δ_x.

If μ is *not* singly atomic, there exists a measurable set $E \subset K$ such that $0 < \mu(E) < 1$. Then

$$\mu = \mu(E)[\frac{1}{\mu(E)}\mu|_E] + (1-\mu(E))[\frac{1}{\mu(E^c)}\mu|_{E^c}]$$

is a way to write μ as a convex combinations of other probability measures. □

Problem 6. *Let $\ell^2(\mathbb{Z})$ denote the real Hilbert space of square-summable functions on the integers. Let x_k ($k \geq 1$) be a sequence in $\ell^2(\mathbb{Z})$ that converges coordinate-wise to zero. i.e., such that $\lim_{k\to\infty} x_k(n) = 0$ for all $n \in \mathbb{Z}$.*

(a) Must x_k converge in norm to 0 as $k \to \infty$? What about if $\|x_k\|$ is assumed to be bounded?

(b) x_k converge weakly to 0 as $k \to \infty$? What about if $\|x_k\|$ is assumed to be bounded?

Justify your answers (by proof or counter-example).

Proof. (a) No to both. The counterexample in both cases is the sequence $(x_k) = (\delta_k)_{k\in\mathbb{N}}$, where

$$\delta_n(z) = \begin{cases} 1 & n = z \\ 0 & n \neq z \end{cases}.$$

The ℓ^2 norm of each of these functions is 1, and $x_k(n)$ is eventually zero for all n.

(b) If $\|x_n\|$ is not bounded, this is untrue. We may consider $g(n) = \delta_{\{m:m\geq 1\}}(n)\frac{1}{n}$ and take $x_k(n) = n\delta_k(n)$. Once again, each $x_k(n)$ is eventually zero, but $\int x_k g = 1$ for all k. Now fix $\varepsilon > 0$ and $g \in \ell^2(\mathbb{Z})$. Pick N such that $\int_{|n|>N} |g|^2 < \varepsilon$. Further pick a K such that $k > K$ implies $\sup_{n\in[-N,N]} |x_k(n)| < \frac{\varepsilon}{2N+1}$. Then for $k > K$,

$$\int_{\mathbb{Z}} |x_k g| = \int_{|n|\leq N} |x_k(n)g(n)| + \int_{|n|>N} |x_k(n)g(n)| < \varepsilon(\sup_n\{|g(n)|\} + \|x_k(n)\|_2).$$

So if $\|x_k\|$ is bounded, we may replace $\sup_n \|x_k(n)\|$ with a uniform bound M. So x_n converges weakly to 0. □

Problem 7. *Let X be a second countable (that is, having a countable basis of open sets) and normal topological space. Show that there is a countable family $\mathcal{F}$ of continuous functions from X into the interval $[0,1]$ that separates points and closed sets: i.e., such that if $x \in X$ and C is closed subset of X with $x \notin C$, then there is $f \in \mathcal{F}$ such that $f(x) = 0$ and $f(C) \subseteq \{1\}$.*

Proof. Let $(B_i)_1^\infty$ be a basis for X. If $B_i \subset \overline{B_i} \subset B_j$ for some $i, j \in \mathbb{N}$, then $\overline{B_i}$, B_j^c are disjoint closed sets, and Urysohn's lemma yields a function $f_{ij} \in C(X,[0,1])$ such that $f_{ij}(\overline{B_i}) \subset \{0\}$, $f_{ij}(B_j^c) \subset \{1\}$.

Let $\mathcal{F} := \{f_{ij} : (i,j) \in \mathbb{N}^2$ when $B_i \subset \overline{B_i} \subset B_j\}$. It is clear that $\mathcal{F}$ is countable; we need to show $\mathcal{F}$ separates points and closed sets.

Let $x \in X$ and $C \subset X$ be closed with $x \notin C$. By definition of basis, there exists some B_j such that $x \in B_j \subset C^c$. Thus by normality there are two disjoint open sets U, V such that $x \in U$ and $B_j^c \subset V$. Again by definition of basis, there is some B_i such that $x \in B_i \subset U$. Now

$$B_i \subset \overline{B_i} \subset \overline{U} \subset V^c \subset B_j.$$

Hence there is some $f_{ij} \in \mathcal{F}$ such that $f_{ij}(\overline{B_i}) = \{0\}$ and $f_{ij}(B_j^c) = \{1\}$. Since $x \in B_i$ and $C \subset B_j^c$, we are done. □

Problem 8. *Let $f \in L^1(0,\infty)$ and define*

$$h(x) = \int_0^\infty (x+y)^{-1} f(y)\,dy$$

for $x > 0$. Show that h is differentiable at all $x > 0$ and show $h' \in L^1(r,\infty)$ for every $r > 0$. What about for $r = 0$? (Justify your answer.)

Proof. The derivative h' can be shown to be

$$h'(x) = \int_0^\infty -\frac{1}{(x+y)^2} f(y)\,dy$$

in a number of ways. We use the classical derivation formula.

$$\lim_{k\to 0} \frac{h(x+k)-h(x)}{h} = \lim_{k\to 0} \int_0^\infty \frac{-1}{(x+y+k)(x+y)} f(y)\,dy.$$

If we let k decrease to 0, these functions are monotonically increasing, and if k increases to 0 it is monotonically decreasing. In particular, when $k = -\frac{x}{2}$,

$$\int \left| \frac{-1}{(x+y+k)(x+y)} f(y) \right| \leq \left| \frac{2}{x^2} \right| \int |f(y)| < \infty,$$

so MCT applies and gives us the desired formula.

We now want to calculate the L^1 norm of h' on (r, ∞) for $r > 0$. We use Tonelli below to switch the order of integration:

$$\begin{aligned}\int_{r'}^{\infty} \int_0^{\infty} |\frac{1}{(x+y)^2} f(y)|\, dy\, dx &= \int_0^{\infty} |f(y)| \int_r^{\infty} \frac{1}{(x+y)^2}\, dx\, dy \\ &= \int_0^{\infty} |f(y)||\frac{1}{r+y}|\, dy \\ &=\leq \frac{1}{r} \int_0^{\infty} |f(y)| < \infty.\end{aligned}$$

So $h' \in L^1(r, \infty)$ for $r > 0$. The problem of whether $h' \in L^1(0, \infty)$ is undetermined when $r = 0$, as one needs more information to determine whether

$$\frac{|f(y)|}{y}$$

is integrable from the right at $y = 0$. □

Problem 9. *Suppose X is a Banach space and Y is a normed linear space and $T : X \to Y$ is a linear map such that for every bounded linear functional $g \in Y^*$ we have $g \circ T$ is bounded. Show that T is bounded.*

Proof. Compare this to Exercise 37 in Chapter 5 of Folland. There are several different ways to solve this problem - I believe one uses the closed graph theorem. Here is my favorite solution.

This problem can be rephrased as saying ($T^*(g) := g \circ T$ is bounded for all $g \in Y^*$) implies (T is bounded). Let $x \mapsto \hat{x}$ be the inclusion map of X into X^{**}. Note

$$\sup_{\|x\|=1} \|\widehat{Tx}(f)\| = \sup_{\|x\|=1} \|(f \circ T)(x)\| = \|f \circ T\| < \infty.$$

This is true pointwise for each $f \in Y^*$. Hence by Uniform Boundedness

$$\sup_{\|x\|=1} \|\widehat{Tx}\| = \sup_{\|x\|=1} \|Tx\| < \infty$$

since $\cdot \mapsto \hat{\cdot}$ is an isometry. But this means exactly that T is bounded. □

Problem 10. *Let X be a real Banach space and suppose C is a closed subset of X such that*

1. *$x_1 + x_2 \in C$ for all $x_1, x_2 \in C$;*
2. *$\lambda x \in C$ for all $x \in C$ and $\lambda > 0$; and*
3. *for all $x \in X$ there exist $x_1, x_2 \in C$ such that $x = x_1 - x_2$.*

Prove that, for some $M > 0$, the unit ball of X is contained in the closure of

$$\{x_1 - x_2 | x_i \in C, \|x_i\| \leq M (i = 1, 2)\}.$$

Deduce that, for some $K > 0$, every $x \in X$ can be written as $x = x_1 - x_2$, with $x_i \in C$ and $\|x_i\| \leq K\|x\|$. (In fact, any $K > M$ will do, but you need not show this.)

Proof. This set C could be considered the definition of a "positive cone" - i.e., the elements of C are the "positive part" of the set X. If $X = \mathbb{R}^n$, then $C = \{(a_i)_{i=1}^n : a_i \geq 0, i \in [n]\}$ satisfies these conditions.

Let's define

$$A_K := \{x_1 - x_2 : x_i \in C, \|x_i\| \leq K\}$$

and note that, by condition (iii),

$$X = \bigcup_{K=1}^{\infty} A_K$$

We claim that, if the problem statement were false, A_K (which are closed since C is closed and the norm is continuous) has empty interior for all K. Say A_K contains some $B(\varepsilon, x)$. Let $x = x_1' - x_2'$ for some $x_i' \in C$, $\|x_i'\| \leq K$. Then

$$B(\varepsilon, 0) \subset A_K - x = \{(x_1 - x_1') - (x_2 - x_2') : x_i \in C, \|x_i\| \leq K\} \subset A_{2K}.$$

Furthermore,

$$B(1, 0) \subset \frac{1}{\varepsilon} A_{2K} \subset A_{\text{ceil}(\frac{2K}{\varepsilon})}.$$

So the A_K would have empty interior. But in this case these are nowhere dense sets, so by Baire Category Theorem we are formced to conclude the problem statement is true.

The second part of the statement is obvious, as for any $x \in X$, $\frac{x}{2\|x\|} \in B(1, 0)$. So $K = 2M$ will suffice. □

31 January 2009

Problem 1. *Let $F \subset \mathbb{R}^n$ be compact and prove that the convex hull $\text{conv}(F)$ is compact. (You may use without proof the theorem of Carathéodory that states that every point in the convex hull of any subset S of $\mathbb{R}^n$ is a convex combination of $n + 1$ or fewer points of S.)*

Proof. We first prove a lemma:

Lemma. *Let A be compact and let $(a_i^k)_{k \in \mathbb{N}}$ be sequences in elements of A for $i \in [n]$. Then there exists an infinite subset $K \subset \mathbb{N}$ such that $(a_i^k) \overset{k \to \infty}{\to} a_i \in A$ for each i.*

Proof. This is an induction argument on the number of sequences n. For $n = 1$ this is true by the definition of compactness. Assume the argument is true for $n = n_0$ and let $(a_i^k)_k$ be $n_0 + 1$ sequences in A. If K_{n_0} is the guaranteed infinite subset as stated above for (a_i^k) for $i \in [n_0]$, then by definition of compactness again there is a further subset $K_{n_0+1} \subset K_{n_0}$ such that $(a^k_{n_0+1})_k$ converges in A as well; this is the desired subset. $\square$

This lemma will do the heavy lifting for us moving forward. Let $x_k = \sum_{i=1}^{n+1} \alpha_i^k f_i^k$ be a sequence in $\text{conv}(F)$; it suffices to show that x_k converges to some $x \in \text{conv}(F)$ (since $\mathbb{R}^n$ is second countable, sequential compactness is sufficient, although one can make a similar argument using nets and subnets). Then by the lemma there exists an infinite subset $K \subset \mathbb{N}$ such that $(\alpha_i^k)_k$ converges for all $i \in [n+1]$; say $\alpha_i^k \overset{k\to\infty}{\to} \alpha_i \in \mathbb{R}$. Then certainly $\alpha_i \in [0,1]$ for all i since $[0,1]$ is compact, and also

$$\sum_{i=1}^{n+1} \alpha_i = \sum_{i=1}^{n+1} \lim_k \alpha_i^k = \lim_k \sum_{i=1}^{n+1} \alpha_i^k = 1.$$

Similarly by the lemma we can find a further infinite subset $L \subset K$ such that $(f_i^k)_k$ converges for all $i \in [n+1]$. If we set f_i to be this convergent point for each i, then since F is compact $f_i \in F$. Since α_i is bounded in norm we have the sequence $(\alpha_i^k f_i^k)_k \to \alpha_i f_i$, and similarly $(x_k) = (\sum_{i=1}^{n+1} \alpha_i^k f_i^k)_k \to \sum_{i=1}^{n+1} \alpha_i f_i$, an element in $\text{conv}(F)$. $\square$

Problem 2. *Let $(X, \mathcal{M}, \rho)$ be a finite measure space. Suppose $\mathfrak{U} \subset \mathcal{M}$ is an algebra of sets and $\mu : \mathfrak{U} \to \mathbb{C}$ is a complex, finitely additive measure such that $|\mu(E)| \leq \rho(E) < \infty$ for all $E \in \mathfrak{U}$. Show that there is a complex measure $\nu : \mathcal{M} \to \mathbb{C}$, whose restriction to $\mathfrak{U}$ is μ, and such that $|\nu(E)| \leq \rho(E)$ for all $E \in \mathcal{M}$. (Hint: you may want to consider the set of simple functions of the form $\sum_1^n c_i 1_{E_i}$.)*

Proof. Although this is a problem about measures, the wording actually suggests approaching this using the Hahn-Banach theorem (note that we want to maintain some upper bound on a measure, possibly one we can derive from an extended linear functional). To this end, let's define

$$A := \{\sum_{i=1}^n c_i 1_{E_i} : n \in \mathbb{N}, c_i \in \mathbb{C}, E_i \in \mathfrak{U} \text{ disjoint}\}.$$

This is a subspace of $L^1(\rho)$ since $(X, \mathcal{M}, \rho)$ is a finite measure space. (It is of course not necessarily closed, but this does not matter to us.) Let us define a linear functional on A

$$f : \sum_1^n c_i 1_{E_i} \mapsto \sum_1^n c_i \mu(E_i)$$

to be "integration over μ". Clearly f is a linear functional. Also,

$$\begin{aligned} |f(\sum_1^n c_i 1_{E_i})| &\leq \sum_1^n |c_i||\mu(E_i)| \\ &\leq \sum_1^n |c_i|\rho(E_i) \end{aligned}$$

by assumption. This later term is in fact $\|c_i 1_{E_i}\|_{L^1(\rho)}$, so if this simple function is denoted as ϕ we have $|f(\phi)| \leq \|\phi\|_1$. Note that $\|\cdot\|_1$ is a sublinear functional.

Now we may apply Hahn-Banach. We get a linear functional F on *all* of $L^1(\rho)$ such that $|F(\cdot)| \leq \|\cdot\|_1$. We note that this implies F is continuous (since $\|\cdot\|_1$ is continuous).

Define $\nu : \mathcal{M} \to \mathbb{C}$ to be

$$\nu(E) := F(1_E).$$

Clearly $\nu|_{\mathfrak{U}} = \mu$. If $(E_j) \subset \mathcal{M}$ is disjoint, then

$$\begin{aligned}
\nu(\bigcup_1^\infty E_j) &= F(\lim_{n\to\infty} 1_{\bigcup_{j=1}^n E_j}) \\
&= \lim_{n\to\infty} F(1_{\bigcup_{j=1}^n E_j}) \\
&= \lim_{n\to\infty} F(\sum_{j=1}^n 1_{E_j}) = \lim_{n\to\infty} \sum_{j=1}^n F(1_{E_j}) \\
&= \sum_{j=1}^\infty \nu(E_j).
\end{aligned}$$

To verify this first equality, it is sufficient to see $1_{\bigcup_{j=1}^\infty E_j}$ has finite $L^1(\rho)$ norm and use continuity from below, and this is straightforward since ρ itself is a finite measure. □

Problem 3. *Given* $p \in [1, \infty)$ *and* $f \in L^p([0, \infty))$, *prove*

$$\lim_{n\to\infty} \int_0^\infty f(x)e^{-nx}\,dx = 0.$$

Proof. You already know what it is! We want to use Lebesgue Dominated Convergence Theorem (LDCT) to move the limit inside the integral. To do this we use a Hölder inequality on the dominating function fe^{-x}:

$$\|f(x)e^{-x}\|_1 \leq \|f\|_p \|e^{-x}\|_q$$

where q is the conjugate exponent to p (in particular, $q \geq 1$), and

$$(\int_0^\infty e^{-qx})^{1/q} = (-\frac{1}{q}[e^{-qx}]_0^\infty)^{1/q} = (\frac{1}{q})^{1/q} \leq 1.$$

Hence $fe^{-x} \in L^1([0, \infty))$, and

$$\lim_{n\to\infty} \int_0^\infty f(x)e^{-nx}\,dx = \int_0^\infty \lim_{n\to\infty} f(x)e^{-nx}\,dx = \int_0^\infty \delta_{x=0} f(0)\,dx = 0.$$

□

Problem 4. *For each bounded, real-valued, Lebesgue measurable function* f *on* $[0, 1]$, *prove that the sets*

$$\begin{aligned}
U(f) &:= \{(x, y)|x \in [0, 1], y \geq f(x)\}, \\
L(f) &:= \{(x, y)|x \in [0, 1], y \leq f(x)\}, \\
G(f) &:= \{(x, f(x))|x \in [0, 1]\}
\end{aligned}$$

are Lebesgue measurable subsets of $[0,1] \times \mathbb{R}$. *(You may want to consider simple functions first.) Then prove that* $G(f)$ *is a null set (with respect to Lebesgue measure).*

Proof. If $f = \sum_1^n c_i 1_{E_i}$ is a simple function, $U(f)$ [resp. $L(f)$] is equal to $\bigcup_1^n E_i \times [c_i, \infty)$ [resp. $\bigcup_1^n E_i \times (-\infty, c_i]$], which are clearly Lebesgue measurable as each factor is. For arbitrary bounded Lebesgue-measurable functions $f : [0,1] \to \mathbb{R}$, we then apply Theorem 2.10 to get an increasing sequence ϕ_n of simple functions converging uniformly to f and take the intersection of $U(\phi_n)$ to get $U(f)$. Note $U(f) = L(-f)$, so this shows $U(f)$ and $L(f)$ are Lebesgue measurable.

Now $G(f) = U(f) \cap L(f)$ is also Lebesgue measurable. To show $G(f)$ is a null set, we evaluate $(m \times m)(G(f)) = \int \int 1_{G(f)}\, dy\, dx$ (we may do this since $1_{G(f)} \in L^+$ (using Lebesgue measurability of $G(f)$!) and by invoking Tonelli). We write

$$\int \int 1_{G(f)}(x,y)\, dy\, dx = \int_{[0,1]} m(\{f(x)\})\, dx = \int_{[0,1]} 0\, dx = 0.$$

□

Problem 5. *Let* $\phi : C_0(\mathbb{R}) \to \mathbb{C}$ *be a bounded linear functional and suppose* μ *is a complex Borel measure on* $\mathbb{R}$ *such that* $\phi(f) = \int f\, d\mu$ *for every rational function* f *over the field of complex numbers whose restriction to* $\mathbb{R}$ *belongs to* $C_0(\mathbb{R})$. *Show that the formula* $\phi(f) = \int f\, d\mu$ *holds for all* $f \in C_0(\mathbb{R})$.

Proof. The question ought to be a bit more clear: we must take the *real* parts of functions f before applying ϕ to them. Otherwise we would need ϕ to be defined on a bigger set than $C_0(\mathbb{R})$. We will make the former assumption.

Although this may appear at first to be a Riesz representation problem, this is actually a Stone-Weierstrass problem.

Let $\mathcal{A} := \{f = \frac{p}{q}|_{\mathbb{R}} : p, q \text{ polynomials in } \mathbb{C}, f|_{\mathbb{R}} \in C_0(\mathbb{R})\}$. We now go through the sequence of S-W questions:

Is $\mathcal{A}$ is an algebra? Yes, since the space of rational functions *and* $C_0(\mathbb{R})$ are both algebras - and since the space of rational functions is closed under multiplicative inverses. It is also easy to see that $\mathcal{A}$ is closed under taking complex conjugates.

Does $\mathcal{A}$ separate points of $\mathbb{R}$? We only need the polynomials with real coefficients to do that. Since the polynomial $(x - x_0 + 1)$ evaluates to 1 when $x = x_0$ for any $x_0 \in \mathbb{R}$, the closure of $\mathcal{A}$ in $C_0(\mathbb{R})$ is indeed all of $C_0(\mathbb{R})$.

Now for any $f \in C_0(\mathbb{R})$ we can find $f_n \in \mathcal{A}$ converging to f uniformly. Since ϕ is bounded, it is continuous, so

$$\phi(f) = \lim \phi(f_n) = \lim \int f_n\, d\mu.$$

Since μ is complex and $C_0(\mathbb{R})$ is bounded, we may use LDCT to switch the limit and the integral and get $\phi(f) = \int f\, d\mu$ as desired. This is worth spelling out: there exists an N such that $n > N$ implies $|f_n| \le |f_N| + 1$. Note that this latter function is an $L^1(\mu)$ function:

$$|\int |f_N| + 1\, d\mu| \le |\int |f_N|\, d\mu| + |\mu|(\mathbb{R}),$$

both of which are finite (the first term since ϕ is bounded, the second term by definition of complex measure). This is our desired dominating function. $\square$

Problem 6. *Let T be a surjective linear map from a Banach space X to a Banach space Y satisfying*

$$\|Tx\| \geq \frac{1}{2009}\|x\|$$

for all $x \in X$. Show that T is bounded.

Proof. Note that this norm condition implies that T is injective: if $x \neq y$, then $\|T(x-y)\| \geq \frac{1}{2009}\|x-y\| > 0$, so $Tx \neq Ty$. So T is a bijection between Banach spaces, and the inverse map T^{-1} exists. Note the inverse map of a linear map is also linear. To see this, let $y_1 = T(x_1)$, $y_2 = T(x_2)$ for $x_i \in X$ and $y_i \in Y$, $i \in [2]$. Then

$$\begin{aligned} T^{-1}(\lambda y_1 + y_2) &= T^{-1}T(\lambda x_1 + x_2) = \lambda x_1 + x_2 \\ &= \lambda T^{-1}(y_1) + T^{-1}(y_2). \end{aligned}$$

T^{-1} is also bounded; in fact $\|x\| \geq \frac{1}{2009}\|T^{-1}x\| \Rightarrow \|T^{-1}x\| \leq 2009\|x\|$. So T^{-1} is a bounded linear map from Y to X, so as a consequence of the Open Mapping Theorem (Corollary 5.11) T^{-1} is an isomorphism. Hence T is bounded. $\square$

Problem 7. *Let X be an infinite-dimensional Banach space. Show*

(a) the unit ball $\{x \in X | \|x\| \leq 1\}$ is closed in the weak topology on X,

(b) every nonempty, weakly open subset of X is unbounded, and

(c) the weak topology on X is no the topology of a complete metric on X.

Proof. Note the similarity to Problems 48 and 49 in Chapter 5 of Folland.

(a) Suppose (x_n) is a sequence in X such that $\|x_n\| \leq 1$ for all n and, for any $f \in X^*$, $f(x_n) \to f(y)$ for some $y \in X$. If $\|y\| > 1$, there exists a norm-one functional $f \in X^*$ such that $f(y) = \|y\| > 1$. But this contradicts continuity of f since $|f(x_n)| \leq 1$ for all n.

(b) This same proof extends to show that any norm-bounded set has a norm-bounded weak closure. Let U be a nonempty weakly open set. Note that a basic open set in the weak topology are of the form

$$U(F, x, \varepsilon) := \bigcap_{f \in F} \{y \in X : f(y-x) < \varepsilon\}$$

where $F \subset X^*$ is finite, $x \in X$, $\varepsilon > 0$ (it suffices to consider a single point x since the weakest topology making functions in X^* continuous coincides with the weakest topology making functions in X^* continuous *at every point* $x \in X$). Let's say a basic open set of this form is contained in U. For any $f \in X^*$, since the range of f has dimension ≤ 1, the dimension of $X/N(f)$ is ≤ 1 as well (see Exercise 35 of Chapter 5 in Folland). So $\dim(X/(\bigcap_{f\in F} N(f))) < \infty$. Since X is infinite-dimensional, this implies $\dim(\bigcap_{f\in F} N(f)) = \infty$. We can then find a nonzero $y \in \bigcap_{f\in F} N(f)$. So $x + ny \in U$ (since $f([x+ny]-x) = 0$) for all n, and $\|x+ny\| \to \infty$ as $|n| \to \infty$. So U is unbounded.

(c) In particular, (b) shows that $B(n,0)$ is nowhere dense for $n \in \mathbb{N}$. Yet $X = \bigcup_{n=1}^\infty B(n,0)$. So if X were the topology of a complete metric on X, this would be a direct contradiction with Baire Category Theorem. $\square$

Problem 8. *(a) Let $(X,\mathcal{M},\mu)$ be a measure space and suppose $E_n \in \mathcal{M}$ are such that*

$$\sum_{n=1}^\infty \mu(E_n) < \infty. \tag{1}$$

Show

$$\mu(\limsup_{n\to\infty} E_n) = 0, \tag{2}$$

where $\limsup_{n\to\infty} E_n = \bigcap_{m=1}^\infty \bigcup_{n=m}^\infty E_n$.

(b) Either prove or disprove that the conclusion (2) follows when hypothesis (1) is replaced by

$$\sum_{n=1}^\infty \mu(E_n)^2 < \infty.$$

Proof. (a) With condition (1), we have that $\mu(\bigcup_{n=1}^\infty E_n) \le \sum_{n=1}^\infty \mu(E_n) < \infty$, so setting $F_m := \bigcup_{n=m}^\infty E_n$ we have F_1 has finite measure and (F_m) is decreasing. So the result follows by continuity from above, since $\mu(\bigcup_{n=m}^\infty E_m) \le \sum_{n=m}^\infty \mu(E_m) \to 0$ as $m \to \infty$.

(b) (2) does not follow with this replacement. Let $F_1 = [0,1]$, and if $F_k = [\alpha,\beta]$ let $F_{k+1} = [\beta, \beta + \frac{1}{k+1}]$ for $k \in \mathbb{N}$. Define $E_k = F_k$ mod 1; that is, $E_k = \{x - n_x : x \in F_k,\ n_x \text{ s.t. } x - n_x \in [0,1]\}$. Then $\mu(E_n) = \frac{1}{n}$, $\sum_{n=1}^\infty \mu(E_n)^2 = \sum_{n=1}^\infty \frac{1}{n^2} < \infty$.

Note in particular that $\bigcap_{n=m}^\infty \bigcup_{n=1}^\infty E_n = [0,1]$ for any m, since by construction these sets wrap around the interval $[0,1]$ infinitely many times ($\sum \frac{1}{n} = \infty$). Hence $\mu(\limsup_n E_n) = 1 \neq 0$. $\square$

Problem 9. *Let $K : [0,1]\times[0,1] \to \mathbb{R}$ be continuous. If $f \in L^1([0,1])$, set*

$$(Tf)(x) = \int_0^1 K(x,y) f(y)\, dy$$

for $x \in [0,1]$.

(a) Show $Tf \in C([0,1])$.

(b) Let B be the unit ball of $L^1([0,1])$ and show that $T(B)$ is relatively compact in $C([0,1])$.

Proof. Note the similarities between this problem and Chapter 4, Problem 63 of Folland. However, the unit ball of $L^1([0,1])$ is slightly larger than the set given in the problem.

(a) K is a continuous operator on a compact space, so it is uniformly continuous. Take $\varepsilon > 0$ and find $\delta > 0$ such that

$$d((x_1,y_1),(x_2,y_2)) < \delta \Rightarrow |K(x_1,y_1) - K(x_2,y_2)| < \varepsilon.$$

Then whenever $|x_1 - x_2| < \delta$,

$$|Tf(x_1) - Tf(x_2)| \leq \int_0^2 |K(x_1, y) - K(x_2, y)||f(y)|\, dy$$
$$\leq \varepsilon \int |f(y)|\, dy.$$

Since $f \in L^1$, $Tf \in C([0, 1])$.

(b) Note in the proof above the δ we chose gave us the same bound of ε regardless of our choice of f as long as $\|f\|_1 \leq 1$. This gives equicontinuity of $T(B)$. Since $|Tf(x)| \leq \|K\|_\infty \|f\|_1$ by Hölder we find this set to be pointwise bounded as well. Hence Arzela-Ascoli I applies and this set is relatively compact. □

Problem 10. *Let μ be a finite Borel measure on $\mathbb{R}$ that is absolutely continuous with respect to Lebesgue measure and show that for every Borel subset A of $\mathbb{R}$, the map $t \mapsto \mu(A + t)$ is continuous from $\mathbb{R}$ to $[0, \infty)$. (Hint: you might first suppose A is an interval).*

Proof. Define $F(x) := \mu((-\infty, x])$. By results used in deriving the Fundamental Theorem of Calculus (Corollary 3.33), $F' \in L^1$ and $F = \int_{-\infty}^{x} F'(t)\, dt$. Hence for $a < a' \in \mathbb{R}$ close enough,

$$\mu((-\infty, x] + a') - \mu((-\infty, x] + a) = \int_{x+a}^{x+a'} F'(t)\, dt$$

is also small, using continuity of the integral limits for L^1 functions.

We now want to show
$$\{A \in \mathcal{P}(X) : a \mapsto \mu(A + a) \text{ is continuous}\}$$

is a σ-algebra. It must be closed under finite (disjoint) unions, and by finiteness of the measure one can approximate the measure of $\mu(\bigcup_{i=1}^{\infty} A_i + a)$ by $\mu(\bigcup_{i=1}^{n} A_i + a)$, so this collection is closed under countable unions as well. Likewise it is closed under complements since $\mu(\mathbb{R}) = \mu(A + a) + \mu(A^c + a)$ is constant. So this collection is a σ-algebra containing the open one-sided rays, so it contains the Borel σ-algebra. □